Coláiste na hOllscoile Corcaigh, Éire
University College Cork, Ireland

CS4618 Artificial Intelligence I

Today: Asymmetric Mutations
Crossover

Thomas Jansen

November 23rd

Announcements

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Fact some asked for a tutorial/review of CS4618 before the in-class test

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Reminder in-class test next Friday
look for sample paper on course web site
late next Monday

Plans for Today

1. PLATEAU
   - Reminder
   - Result for Shifted PLATEAU
2. Crossover
   - Introduction
   - Algorithm with Crossover
3. 1-Point Crossover
   - Idea and Example Function
   - Result
4. Uniform Crossover
   - Idea and Example Function
5. Summary
   - Summary & Take Home Message

Remember: Asymmetric Mutations

Standard Bit Mutations

```
1. For $i \in \{1, 2, \dots, n\}$
2.     With probability $1/n$
3.     set $y[i] := 1 - x[i]$
4.     else set $y[i] := x[i]$
```

(1+1) EA

Asymmetric Mutations

```
1. For $i \in \{1, 2, \dots, n\}$
2.     If $x[i] = 1$
3.     then $p_m = 1/(2\,|x|)$
4.     else $p_m = 1/(2(n - |x|))$.
5.         With probability $p_m$
6.         set $y[i] := 1 - x[i]$
7.         else set $y[i] := x[i]$
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asymmetric (1+1) EA

PLATEAU

Def.

$$\textsc{Plateau}\colon \{0,1\}^n \to \{0,1,\ldots,n+1\}$$

$$\textsc{Plateau}(x) = \begin{cases} n+1 & \text{if } x = 1^n \\ n & \text{if } x = 1^i 0^{n-i},\ i \in \{0,1,\ldots,n-1\} \\ n - |x| & \text{otherwise} \end{cases}$$

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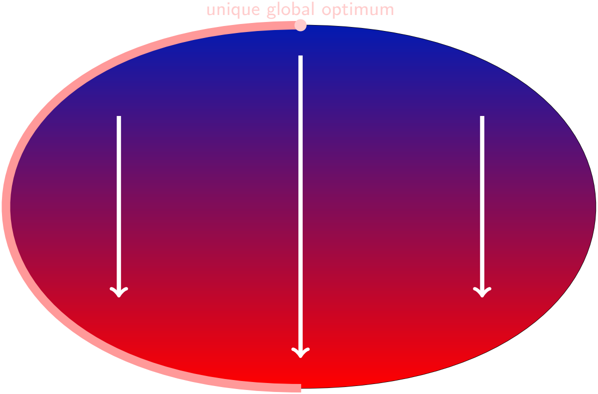

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- $\mathrm{E}\left(T_{\text{(1+1) EA},\text{PLATEAU}}\right) = \Theta\left(n^3\right)$
  $\widehat{=}$efficient
- $\mathrm{Prob}\left(T_{\text{asym. (1+1) EA},\text{PLATEAU}} = n^{o\left(n^{1/6}\right)}\right) = 2^{-\Omega\left(n^{1/6}\right)}$
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with $a_{10} = 101010\cdots10$

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Thus $\mathrm{E}\left(T_{\text{asym. (1+1) EA},\textsc{Plateau}_{a_{01}}}\right) = \Theta(n^3)$

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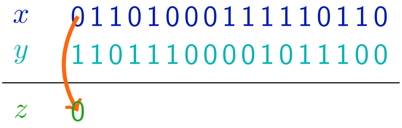

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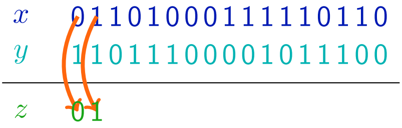

x 0 1 1 0 1 0 0 0 1 1 1 1 1 0 1 1 0

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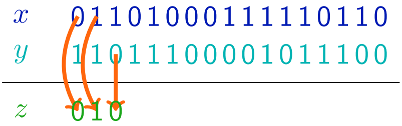

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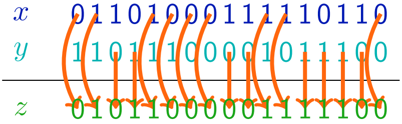

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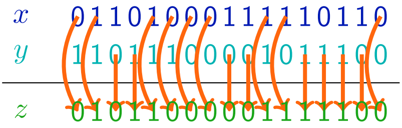

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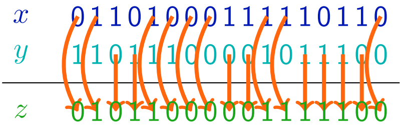

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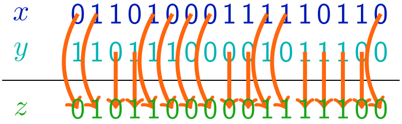

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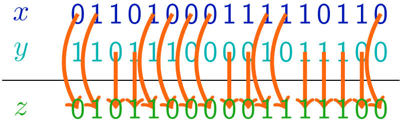

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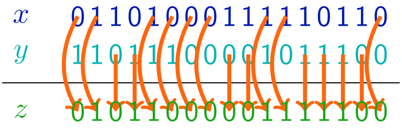

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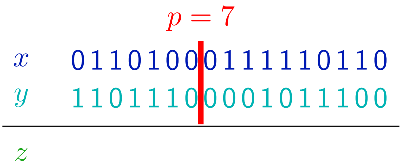

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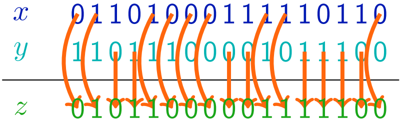

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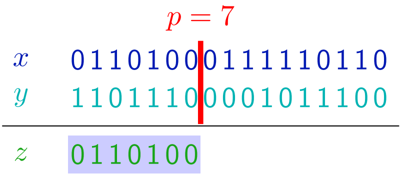

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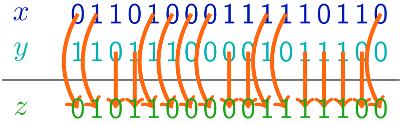

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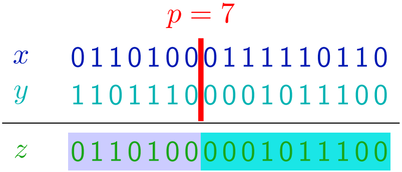

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  - uniform crossover: $\forall x, y\colon\ |\{z \mid z = \mathsf{crossover}(x, y)\}| = 2^{\mathsf{H}(x,y)}$
  - 1-point crossover: $\forall x, y\colon\ |\{z \mid z = \mathsf{crossover}(x, y)\}| \leq n + 1$

On Uniform and 1-Point Crossover

Similarities

- both produce one offspring from two parents
- both respect parents: $\forall i\colon x[i] = y[i] \Rightarrow z[i] = x[i] = y[i]$
- both allow for variability:
  $\forall i\colon x[i] \neq y[i] \Rightarrow 0 < \mathrm{Prob}\,(z[i] = 0) < 1$

Differences

- variability in offspring
  - uniform crossover: $\forall x, y\colon\ |\{z \mid z = \text{crossover}(x, y)\}| = 2^{\mathsf{H}(x,y)}$
  - 1-point crossover: $\forall x, y\colon\ |\{z \mid z = \text{crossover}(x, y)\}| \leq n + 1$
- positional dependencies
  - uniform crossover: $\forall x, y,$ permutations $\sigma\colon$
    $\{z \mid z = \text{crossover}(x, y)\} = \left\{\sigma^{-1}(z) \mid z = \text{crossover}(\sigma(x), \sigma(y))\right\}$

On Uniform and 1-Point Crossover

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Goal demonstrate understanding of uniform and 1-point crossover by presenting appropriate examples where crossover is essential for efficiency

Steady State Genetic Algorithm

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Steady State GA

1. Choose $x_1, x_2, \ldots, x_\mu \in \{0,1\}^n$ uniformly at random.

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2. Repeat
3.     With probability $p_c$ select $z_1, z_2 \in \{x_1, x_2, \ldots, x_\mu\}$ u. a. r.
       $z := \text{crossover}(z_1, z_2)$
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5.     Select new $x_1, x_2, \dots, x_\mu$ out of best $x_1, x_2, \dots, x_\mu, y$.
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7  Until 'decide to stop'
8. Output $x_i$ with $f(x_i) = \max\{f(x_j) \mid 1 \leq j \leq \mu\}$.
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```

Parameters	crossover probability $p_c \in [0,1]$
	population size $\mu \in \mathbb{N}$

Ideas for an Example Function for 1-Point Crossover

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Observation $\forall x, y\colon\ |\{z \mid z = \text{crossover}(x, y)\}| \leq n + 1$

$\Rightarrow$ each possible offspring occurs with good probability

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Observation	$\forall x, y\colon\ \lvert\{z \mid z = \text{crossover}(x, y)\}\rvert \leq n + 1$ $\Rightarrow$ each possible offspring occurs with good probability
Observation	$\mathsf{H}(x, y)$ large $\Rightarrow \min\{\mathsf{H}(x, z), \mathsf{H}(y, z)\}$ may be large

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Remember mutations very sensitive with respect to Hamming distance
'jumps' of size k exponentially unlikely in k
$\operatorname{Prob}\left(\mathsf{H}\left(\text{mutation}(x), x\right) \geq k\right) = e^{-\Omega(k)}$

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$\mathrm{Prob}\left(\mathsf{H}(\mathsf{mutation}(x), x) \geq k\right) = e^{-\Omega(k)}$

Idea construct example function with easy to find local optima
with large Hamming distance to global optimum
where the Hamming distance is easy to bridge for crossover

Example Function for 1-Point Crossover

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Definition $b(x) = \max\{l \mid \exists i\colon x[i] = x[i+1] = \cdots = x[i+l-1] = 1\}$
length of longest block of 1-bits in x

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$b(x) = 4$ 001111, 011110, 111100

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Example $n = 6$, $\text{OneMax}(x) = 4$
$b(x) = 4$ 001111, 011110, 111100
$b(x) = 3$

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$b(x) = 4$ 001111, 011110, 111100
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$b(x) = 2$

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Definition $f_1 \colon \{0,1\}^n \to \mathbb{N}_0$ with

$$f_1(x) = \begin{cases} n^2 + 1 & \text{if } x = 1^n \\ n \cdot |x| + b(x) & \text{if } |x| \leq (2/3)n \\ 0 & \text{otherwise} \end{cases}$$

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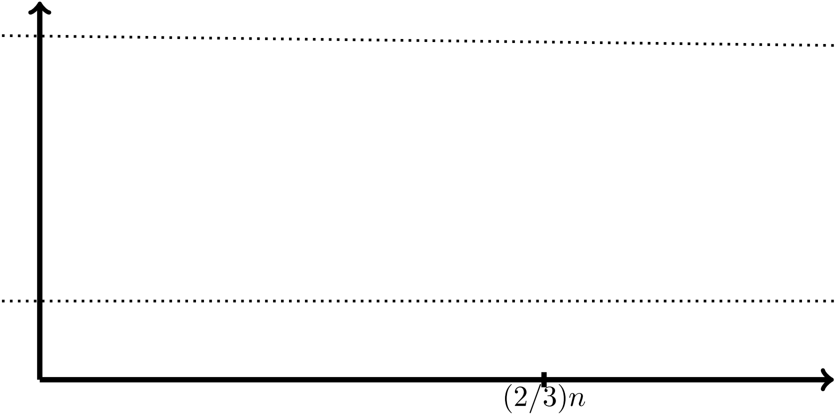

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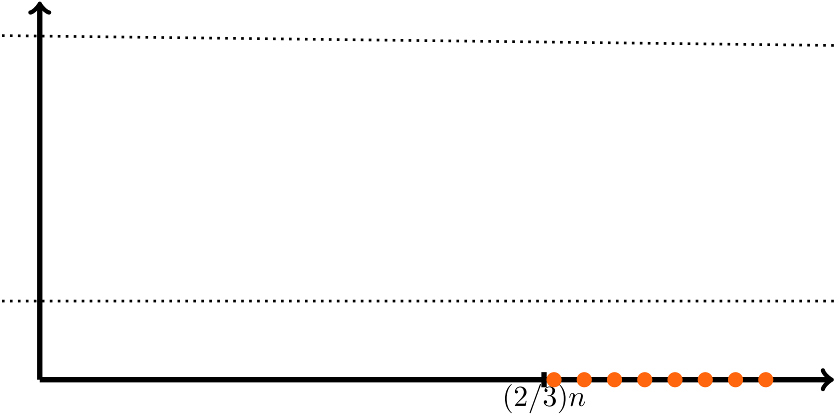

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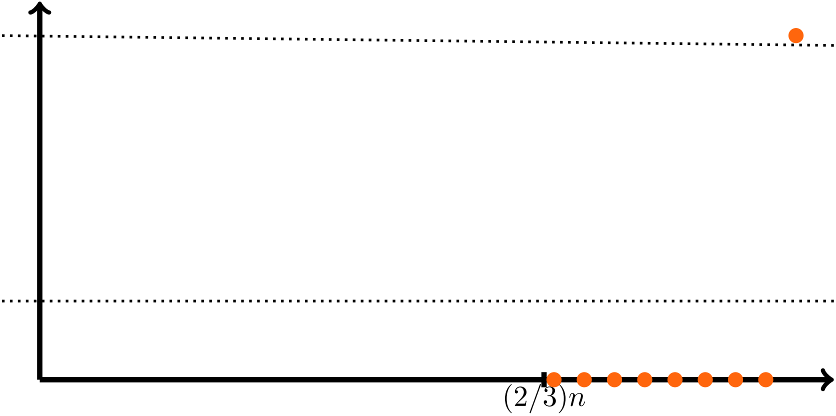

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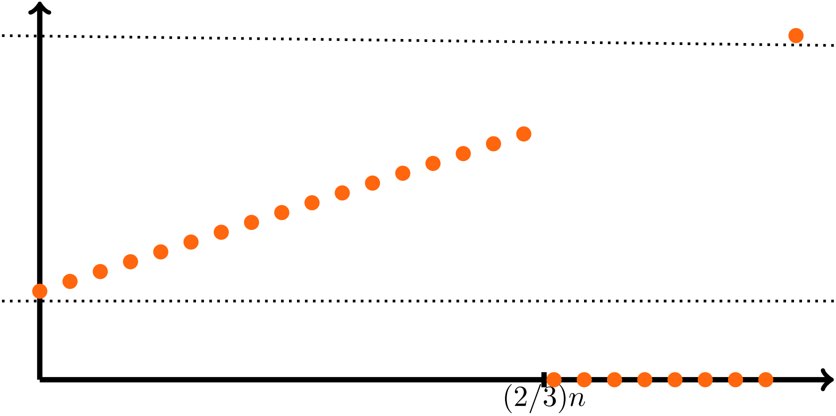

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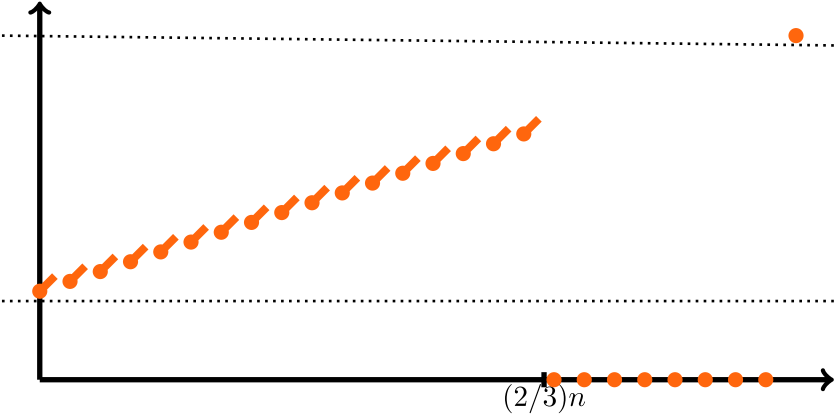

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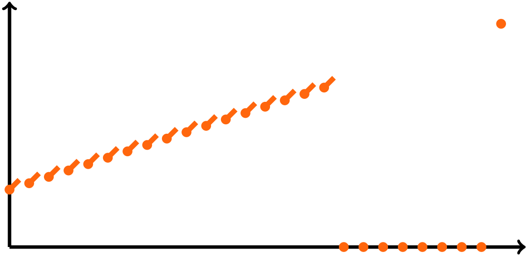

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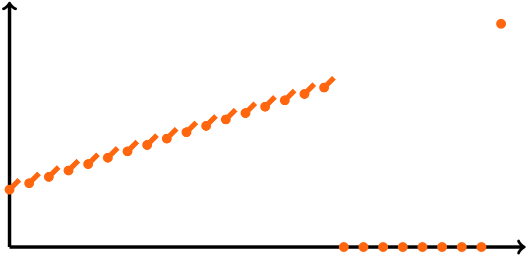

Remember without crossover only mutation remains

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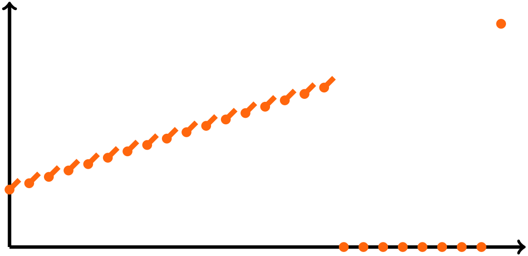

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Observations

- $\mathrm{Prob}\left(\text{initial population left of 'gap'}\right) = 1 - 2^{-\Omega(n)}$

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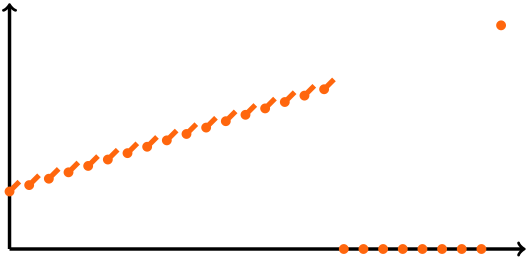


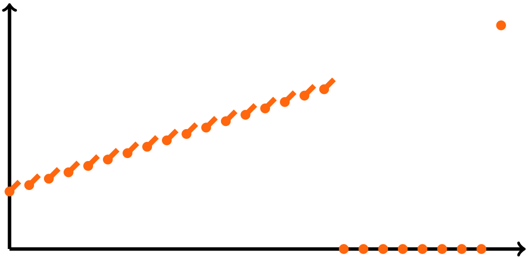
Remember without crossover only mutation remains

Observations

- $\mathrm{Prob}\left(\text{initial population left of 'gap'}\right) = 1 - 2^{-\Omega(n)}$
- simultaneous mutation of $n/3$ bits needed

Without Crossover on f_1

$$f_1(x) = \begin{cases} n^2 + 1 & \text{if } x = 1^n \\ n \cdot |x| + b(x) & \text{if } |x| \leq (2/3)n \\ 0 & \text{otherwise} \end{cases}$$



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- probability for such mutations $2^{-\Omega(n)}$

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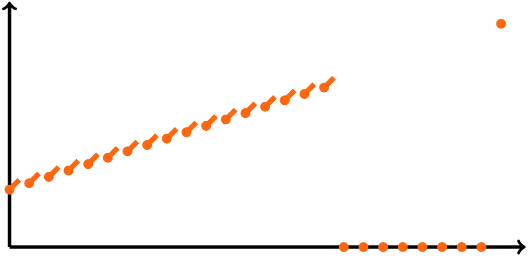

Remember without crossover only mutation remains

Observations

- $\text{Prob}\,(\text{initial population left of 'gap'}) = 1 - 2^{-\Omega(n)}$
- simultaneous mutation of $n/3$ bits needed
- probability for such mutations $2^{-\Omega(n)}$
- $\Rightarrow$ exponential time with overwhelming probability

With 1-Point Crossover on f_1

$$f_1(x) = \begin{cases} n^2+1 & \text{if } x = 1^n \\ n \cdot |x| + b(x) & \text{if } |x| \leq (2/3)n \\ 0 & \text{otherwise} \end{cases}$$

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Theorem

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Thus
$$\mathrm{E}\left(T_{\mathsf{GA},f_1}\right) = O\left(\mu n \log n + \mu n^3 + \mu^2/p_c\right) = O\left(\mu n^3 + \mu^2/p_c\right)$$

☐

Ideas for an Example Function for Uniform Crossover

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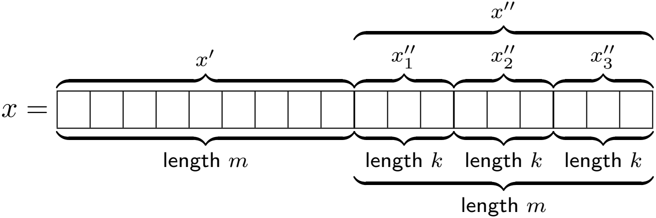

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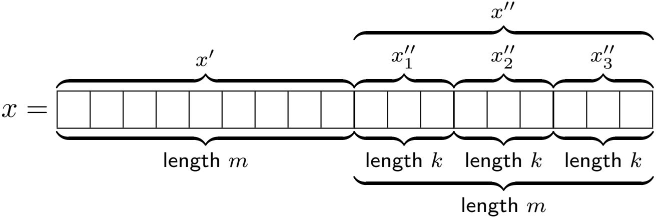

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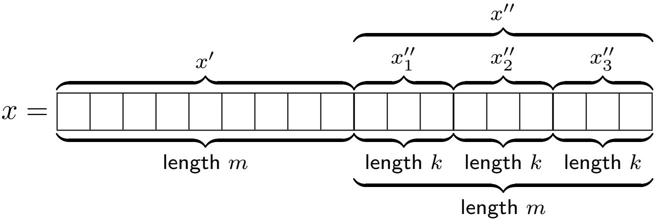

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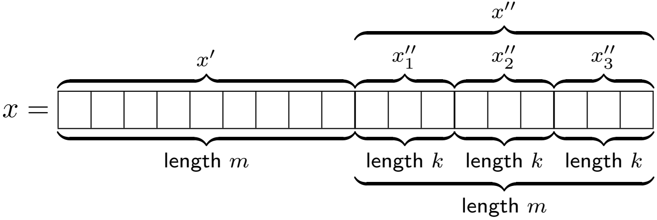

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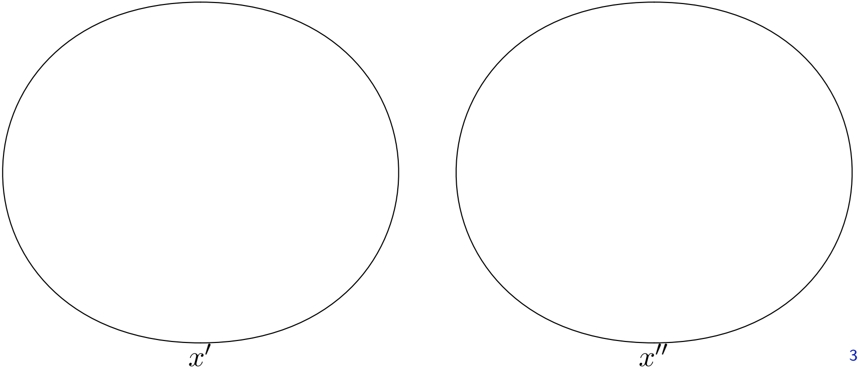

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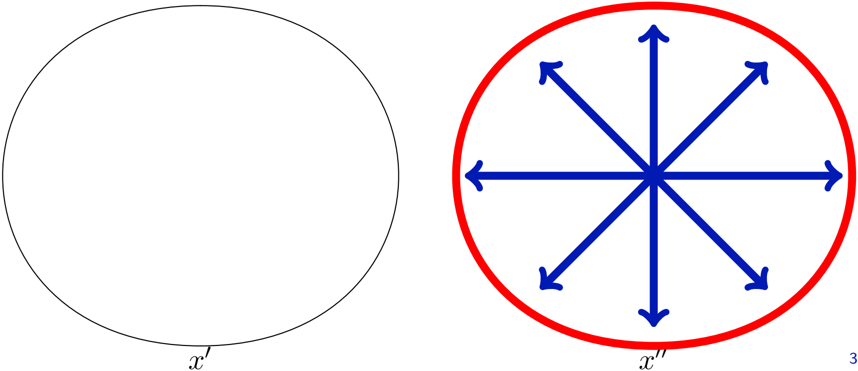

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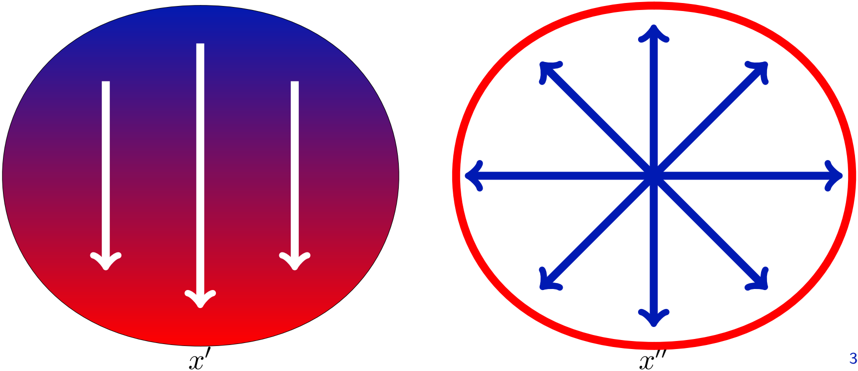

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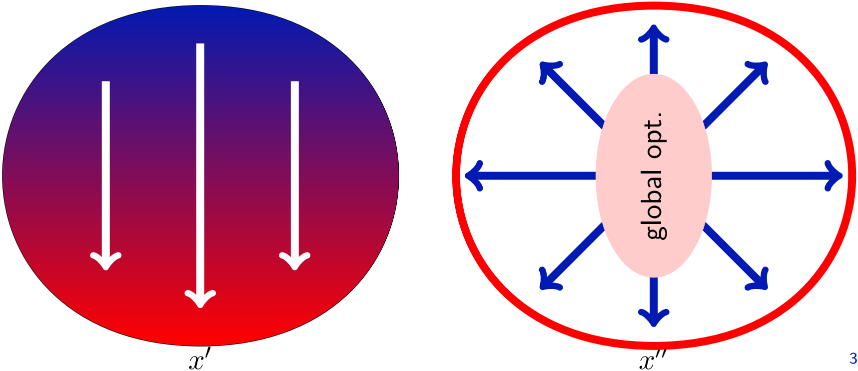

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