

CS4618 Artificial Intelligence I

Today: Analysing and Designing Mutation

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November 21st

Plans for Today

1 Introduction

Motivation

2 Asymmetric Mutations

Asymmetric Mutation Operator

Basic Properties

3 ONEMAX

Result for ONEMAX

Result for Generalised ONEMAX

4 PLATEAU

Result for PLATEAU

5 Summary

Summary & Take Home Message

Thoughts about Applications

Input

Thoughts about Applications

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some nodes



Thoughts about Applications

Input some nodes
costs for connecting two nodes

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Output

connections such that

① for any two nodes \exists path of connections between them



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connections such that

- 1 for any two nodes \exists path of connections between them
- 2 a cheapest set of such connections



Thoughts about Applications

Input some nodes $V = \{v_1, v_2, \dots, v_n\}$
 costs for connecting two nodes $w_{i,j} \in \mathbb{R}^+$

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Add additional restriction 'maximal degree k '
 \rightsquigarrow problem NP-hard
 \rightsquigarrow application of RSH is reasonable

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Caution Know what you are doing!

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2. With probability $1/n$
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 $\leq \frac{1}{k!}$
 $= \Theta\left(1 / \left(\sqrt{k} \cdot (k/e)^k\right)\right)$

Asymmetric Mutations

- $1/(2n) \leq p_m \leq 1/2$
- $E(|\text{mutation}(x)| \mid x)$
 $= |x| \cdot \left(1 - \frac{1}{2|x|}\right) +$
 $(n - |x|) \cdot \frac{1}{2(n-|x|)} = |x|$
- $H(x, y) = 1 \Rightarrow$
 $\text{Prob}(y = \text{mutation}(x)) =$
 $\frac{1}{2l} \left(1 - \frac{1}{2l}\right)^{l-1} \left(1 - \frac{1}{2(n-l)}\right)^{n-l}$
 $= \Theta\left(\frac{1}{l}\right)$
- $\text{Prob}(H(\text{mutation}(x), x) \geq k)$
 $\leq \left(\frac{e^{k-1}}{k^k}\right)$

Properties of Asymmetric Mutations

Standard Bit Mutations

- $p_m = 1/n$
- $E(|\text{mutation}(x)| \mid x)$
 $= |x| \cdot \left(1 - \frac{1}{n}\right) + (n - |x|) \cdot \frac{1}{n}$
 $= |x| + 1 - \frac{2|x|}{n}$
- $H(x, y) = 1 \Rightarrow$
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Expected Number of 1-Bits

Standard Bit Mutations

$$E(|\text{mutation}(x)| \mid x)$$

$$= |x| + 1 - \frac{2|x|}{n}$$

Asymmetric Mutations

$$E(|\text{mutation}(x)| \mid x)$$

$$= |x|$$

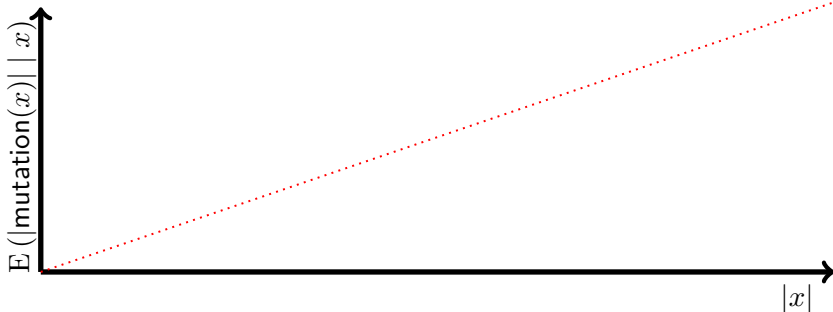
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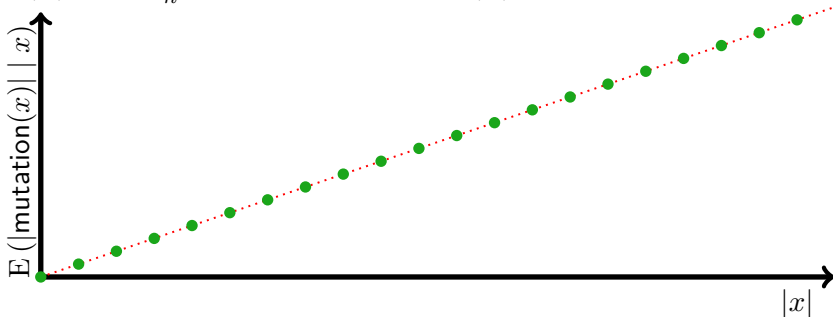
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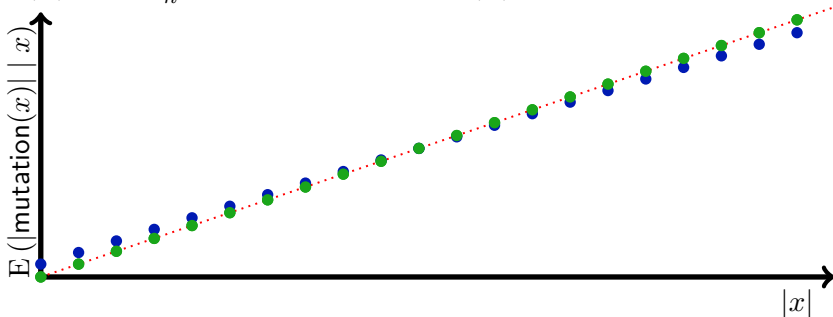
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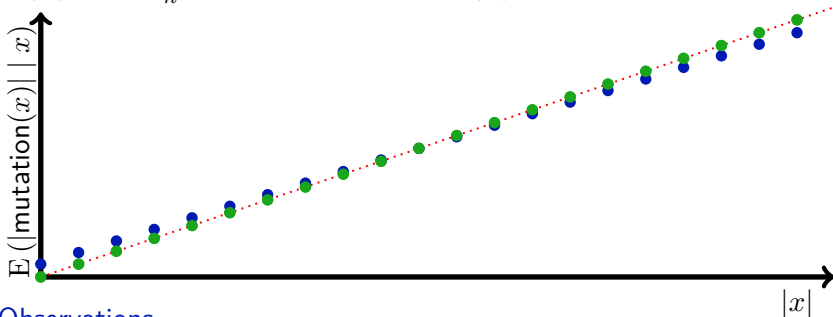
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- standard bit mutations steer towards the middle of the search space

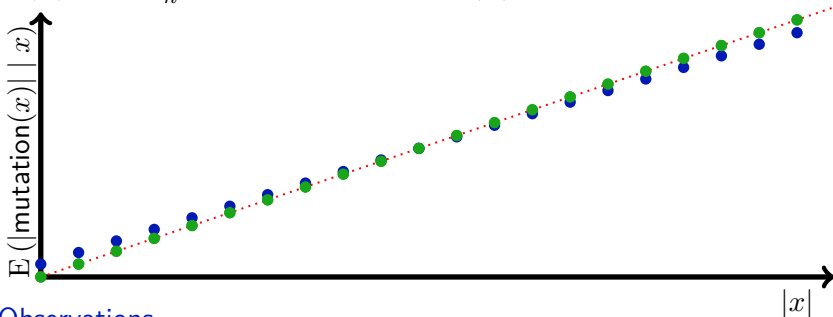
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- standard bit mutations steer towards the middle of the search space
- asymmetric mutations tend not to leave the current level

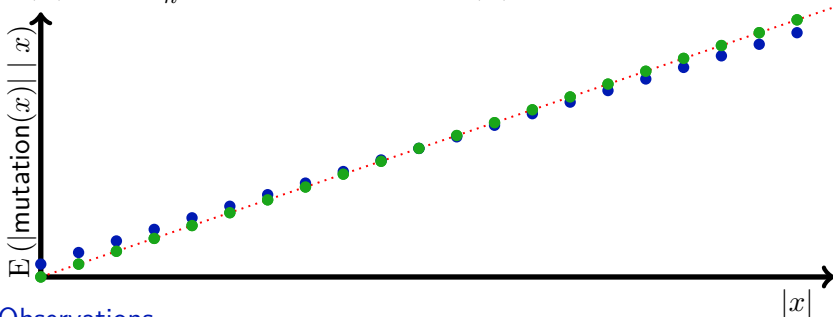
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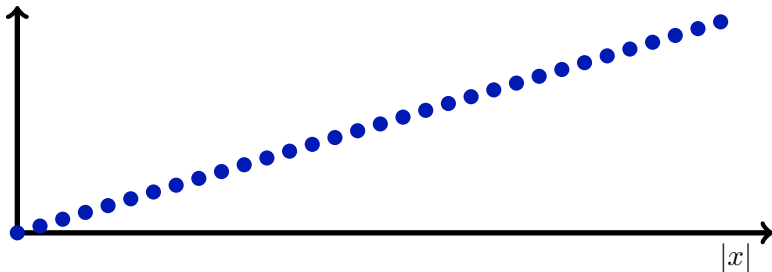


Observations

- standard bit mutations steer towards the middle of the search space
- asymmetric mutations tend not to leave the current level
- asymmetric mutations tend to spend more time at levels with few 0-bits or few 1-bits

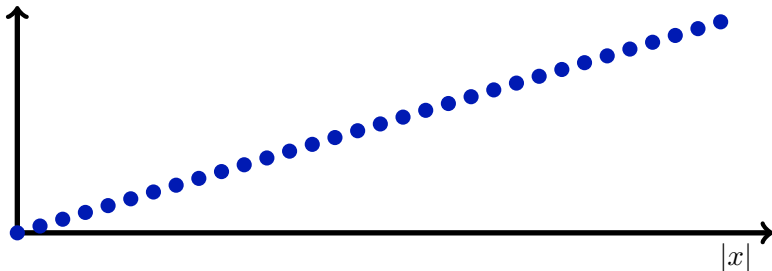
ONEMAX

Remember $\text{ONEMAX}(x) = \sum_{i=1}^n x[i]$



ONEMAX

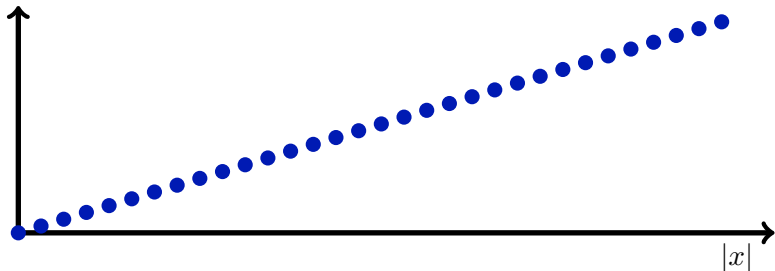
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ONEMAX

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Remember $E(T_{(1+1) \text{ EA, ONEMAX}}) = \Theta(n \log n)$

Theorem

$E(T_{\text{asym. (1+1) EA, ONEMAX}}) = \Theta(n)$

Asymmetric Mutations on ONEMAX: Lower Bound

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Claim $E\left(T_{\text{asym. (1+1) EA, ONEMAX}}\right) = \Omega(n)$

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Technique drift analysis

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Claim $\mathbb{E} \left(T_{\text{asym. (1+1) EA, ONEMAX}} \right) = \Omega(n)$

Technique drift analysis

distance measure $d: \{0, 1\}^n \rightarrow \mathbb{R}_0^+$

expected initial distance $\mathbb{E}(d(x_0))$

maximal expected drift $\Delta = \max \mathbb{E}(d(x_{t-1}) - d(x_t) \mid x_{t-1})$

$\mathbb{E}(T) \geq \mathbb{E}(d(x_0)) / \Delta$

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Discussing the Result on ONEMAX

We have $E\left(T_{\text{asym. (1+1) EA, ONEMAX}}\right) = \Theta(n)$

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- ① from start until $\leq 2(n - |\bar{a}|)$ 0-bits, length T_1
- ② from end of ① until optimum found, length T_2

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Can it get worse?

An Example Function

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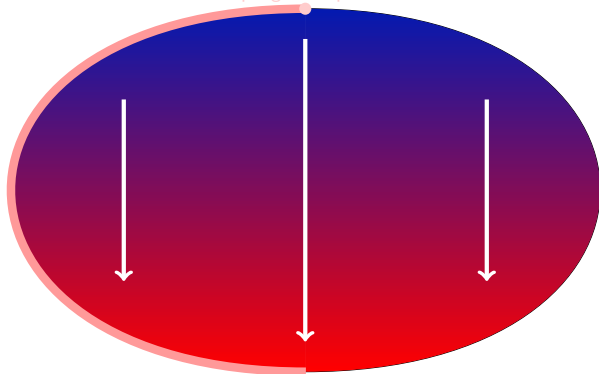
$$\text{PLATEAU}(x) = \begin{cases} n + 1 & \text{if } x = 1^n \\ n & \text{if } x = 1^i 0^{n-i}, i \in \{0, 1, \dots, n - 1\} \\ n - |x| & \text{otherwise} \end{cases}$$

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Is the bias of asymmetric mutations harmful?

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What happens in small mutations with $|x| = \Theta(n^{2/3})$?

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We have $x = 1^i 0^{n-i}$ with $i = \Theta(n^{2/3})$

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Asymmetric Mutations with $|x| = \Theta(n^{2/3})$ on the Plateau

We have $x = 1^i 0^{n-i}$ with $i = \Theta(n^{2/3})$

Observe Prob (increase number of 1-bits)

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$$= 1 - \text{Prob(increase number of 1-bits | change number of 1-bits)}$$

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Asym. Mutations with $|x| = \Theta(n^{2/3})$ on Plateau (cont.)

We have

- Prob (increase number of 1-bits | change number of 1-bits)
 - $= O\left(\frac{1}{n^{1/3}}\right)$
- Prob (decrease number of 1-bits | change number of 1-bits)
 - $= 1 - O\left(\frac{1}{n^{1/3}}\right)$

Asym. Mutations with $|x| = \Theta(n^{2/3})$ on Plateau (cont.)

We have $\text{Prob}(\text{increase number of 1-bits} \mid \text{change number of 1-bits})$
 $= O\left(\frac{1}{n^{1/3}}\right)$
 $\text{Prob}(\text{decrease number of 1-bits} \mid \text{change number of 1-bits})$
 $= 1 - O\left(\frac{1}{n^{1/3}}\right)$

We have unfair random walk
 biased **away** from the global optimum

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 very unlikely (weakly exponential)
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Observation still weakly exponential
 in a weakly exponential number of attempts
 (number of attempts sufficiently small)



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- Incorporating domain knowledge is usually beneficial.

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- Incorporating domain knowledge is usually beneficial.
- Biasing variation can speed search considerably.
- Think and check what you've done.