Coláiste na hOllscoile Corcaigh, Éire
University College Cork, Ireland

CS4618 Artificial Intelligence I

Today: Black-Box Complexity of Unimodal Functions Analysing Mutation

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November 14th

Plans for Today

1. Black Box Complexity of Unimodal Functions
   - Introduction
2. Analysing Mutation
   - Motivation
3. Global Mutations Excel
   - Local Optima
4. Summary
   - Summary & Take Home Message

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- for proof concessions made towards optimal deterministic algorithm
  1. letting it know which functions have probability 0.
  2. giving away for free the knowledge about any p_i with $f(p_i) \leq f(p_j)$ once p_j is sampled,
  3. giving away for free the knowledge about $p_{j+1}, \ldots p_{j+n}$ if p_j is the current known best path point and some point not on the path is sampled,
  4. giving away for free the knowledge about $p_{l(n)}$ (the global optimum) once p_{j+n} is sampled while p_j is the current known

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Considering $i = 1$ and some $j \geq \beta n$ suffices.

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Observations

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4. $H_j \geq (3/5)\gamma n - (2/5)\gamma n$

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We have γn independent random variable $S_t, S_{t+1}, \ldots, S_j \in \{0,1\}$ with $\mathrm{Prob}\left(S_k = 1\right) = 7/10$ and $S := \sum\limits_{k=t}^{j} S_k$.

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$$\mathrm{E}\left(S\right) = (7/10)\gamma n$$

$$\mathrm{Prob}\left(S < \tfrac{3}{5}\gamma n\right)$$
$$= \mathrm{Prob}\left(S < \left(1 - \tfrac{1}{7}\right)\tfrac{7}{10}\gamma n\right)$$

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&= \operatorname{Prob}\left(S < \left(1 - \tfrac{1}{7}\right)\tfrac{7}{10}\gamma n\right) \\
&< e^{-(7/10)\gamma n (1/7)^2/2}
\end{aligned}$$

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$\square$

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We can prove at best lower bound of
$\frac{l'(n)-n+1}{n} > \frac{l(n)}{n^2} - 1 > 2^{n^{\delta}}$.

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Consider random path construction starting in p_i.

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Similar to Lemma
$\mathrm{Prob}\,(\text{hit } x) = 2^{-\Omega(n)}$

Later steps

$N \neq \emptyset$

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Partition N

$N_{\textsf{far}} := \{y \in N \mid \mathsf{H}(y, p_i) \geq \alpha(1/2)n\}$

$N_{\textsf{near}} := N \setminus N_{\textsf{far}}$

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A: path hits x

E: path hits no point in N_{far}

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Clearly, optimal deterministic algorithm avoid N_{far}.

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Thus, we are interested in $\mathrm{Prob}\,(A \mid E)$

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$N_{\mathsf{far}} := \{y \in N \mid \mathsf{H}(y, p_i) \geq \alpha(1/2)n\}$
$N_{\mathsf{near}} := N \setminus N_{\mathsf{far}}$

Case 1 $N_{\mathsf{near}} = \emptyset$

Consider random path construction starting in p_i.

A: path hits x
E: path hits no point in N_{far}

Clearly, optimal deterministic algorithm avoid N_{far}.

Thus, we are interested in $\mathrm{Prob}\,(A \mid E)$
$= \frac{\mathrm{Prob}(A \cap E)}{\mathrm{Prob}(E)} \leq \frac{\mathrm{Prob}(A)}{\mathrm{Prob}(E)}$.

Later steps

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Thus, $\mathrm{Prob}\left(A \mid E\right) \leq \left(1 + 2^{-\Omega(n)}\right) \mathrm{Prob}\left(A\right) = 2^{-\Omega(n)}$.

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Repeat Case 1. $\square$

Mutation Operators

Remember different randomised search heuristics

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- randomised local search
- Metropolis algorithm
- simulated annealing
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$\mathrm{E}(\mathrm{H}(x, y))$

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Observation

local: $\mathrm{H}(x, y) = 1$, $\mathrm{E}(\mathrm{H}(x, y)) = 1$

global: $\mathrm{H}(x, y) \in \{0, 1, \dots, n\}$

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Observation

	local	global
$\mathrm{H}(x, y)$	$= 1$	$\in \{0, 1, \ldots, n\}$
$\mathrm{E}(\mathrm{H}(x, y))$	$= 1$	

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$$\mathrm{H}(x, y) = 1 \qquad \mathrm{E}(\mathrm{H}(x, y)) = 1$$

$$\mathrm{H}(x, y) \in \{0, 1, \ldots, n\} \qquad \mathrm{E}(\mathrm{H}(x, y)) = n \cdot (1/n) = 1$$

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	$\mathrm{H}(x, y) = 1$	$\mathrm{H}(x, y) \in \{0, 1, \ldots, n\}$
	$\mathrm{E}(\mathrm{H}(x, y)) = 1$	$\mathrm{E}(\mathrm{H}(x, y)) = n \cdot (1/n) = 1$

Similar? Perhaps even the same?

An Introductory Example: OneMax

$$\textsc{OneMax}(x) = \sum_{i=1}^{n} x[i]$$

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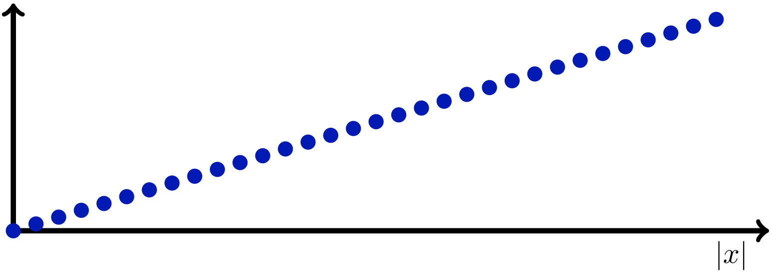

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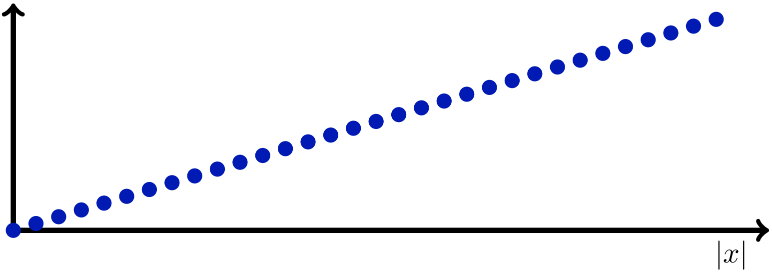

Theorem

$$\mathrm{E}\left(T_{\mathsf{RLS},\textsc{OneMax}}\right) = O(n \log n)$$

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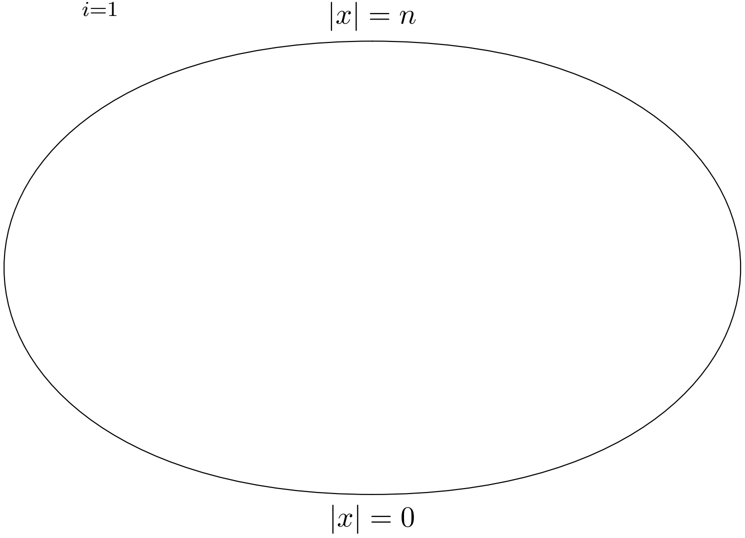

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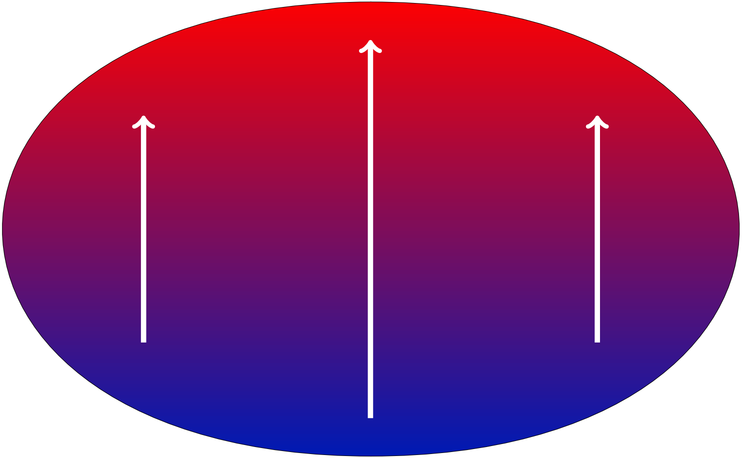
$|x| = n$



$|x| = 0$

RLS on OneMax

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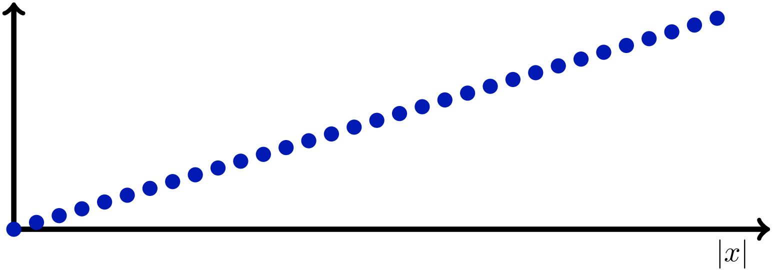
$\square$

The Example OneMaxand the (1+1) EA

$$\textsc{OneMax}(x) = \sum_{i=1}^{n} x[i]$$



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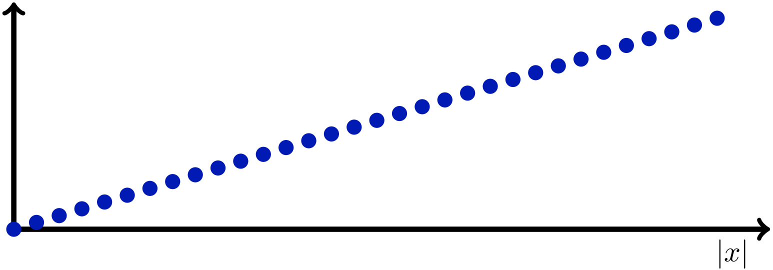

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Interlude: A General Proof Method (Part 1 of 3)

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Definition (Fitness-Based Partitions)

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Remember fitness-based partitions $L_0, L_1, \ldots, L_k$

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the probability for improvement from level L_i.

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Remember fitness-based partitions $L_0, L_1, \ldots, L_k$
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- works without change for RLS
- works for (1+λ) EA if s_i are adapted

Local Optima

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$$f_1(x) = \begin{cases} \textsc{OneMax}(x) & \text{if } |x| \neq n-1 \\ 0 & \text{otherwise} \end{cases}$$

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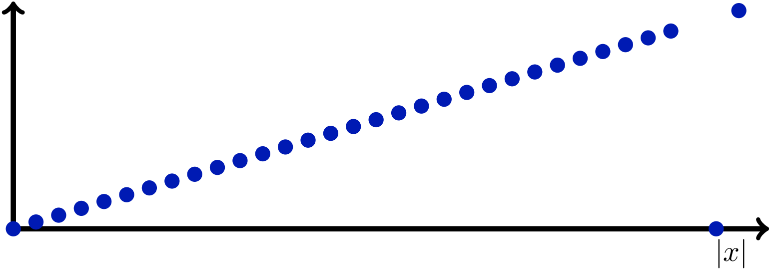

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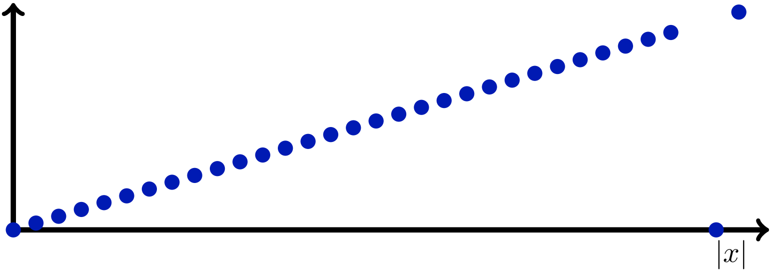

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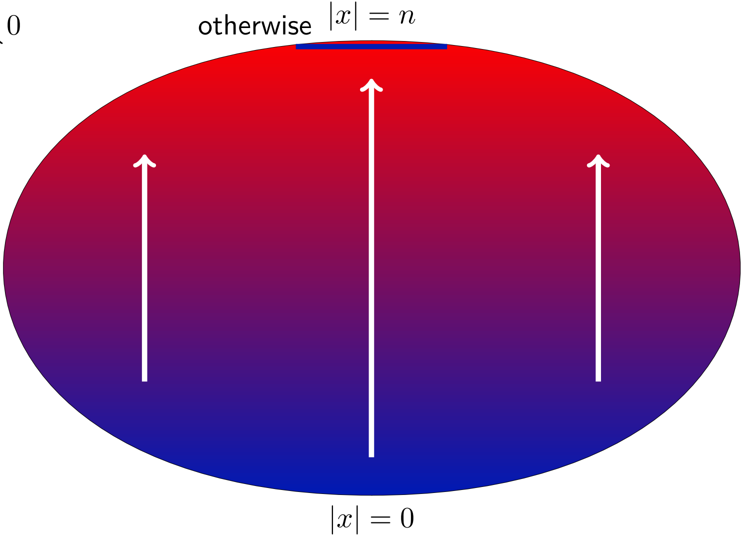

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Thus $\mathrm{E}\left(T_{\text{(1+1) EA},f_1}\right) = O(n \log n) + O(n^2) = O(n^2)$

$\square$

Example function f_1 and RLS

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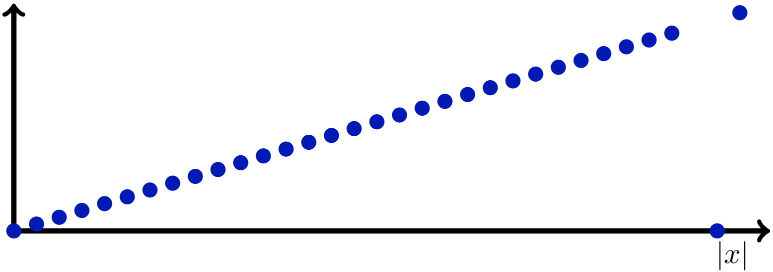

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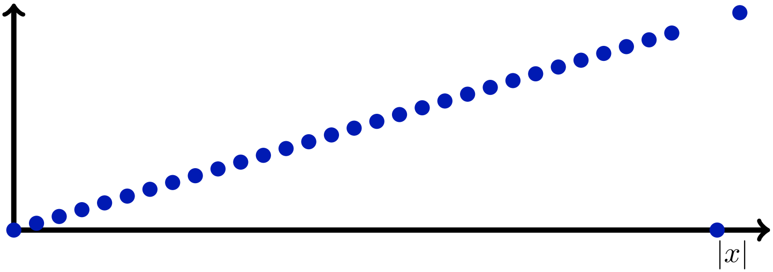

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RLS and the Local Optimum

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- (B) $|x_0| = n-1$ and $|x_1| = n$

Observation

$$\text{Prob}\left(\text{A}\right) = 2^{-n}$$

$$\text{Prob}\left(\text{B}\right) = \text{Prob}\left(|x_0| = n-1\right) \cdot \text{Prob}\left(\text{mutate '}n-1\text{'} \rightsquigarrow \text{'}n\text{'}\right)$$

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Observation

$$\text{Prob}(\text{A}) = 2^{-n}$$

$$\text{Prob}(\text{B}) = \text{Prob}\left(|x_0| = n-1\right) \cdot \text{Prob}\left(\text{mutate } `n-1' \rightsquigarrow `n'\right) = n \cdot 2^{-n} \cdot \frac{1}{n} = 2^{-n}$$

Thus $\text{Prob}\left(\text{RLS avoid local opt.}\right) = \text{Prob}\left(T_{\text{RLS},f_1} < \infty\right)$
$= 2^{-n} + 2^{-n} = 2 \cdot 2^{-n} = 2^{-n+1}$ □

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