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Coláiste na hOllscoile Corcaigh, Éire
University College Cork, Ireland

CS4618 Artificial Intelligence I

Today: Assessment of Randomised Search Heuristics: Black-Box Complexity

Thomas Jansen

November 9^{th}

Plans for Today

Remember

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Randomisation: mixed strategies

row player chooses probability distribution p over rows

column player chooses probability distribution q over columns

Mixed Strategies in Two-Players Zero-Sum Games

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We can have

- randomised algorithms by probability distributions q over $\mathcal{A}$
- probability distributions p over $\mathcal{I}$.

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We get a lower bound for the
worst-case performance of a randomised algorithm by
proving a lower bound on the worst-case performance of an
optimal deterministic algorithm
for an arbitrary probability distribution over the inputs.

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$B_{\text{NEEDLE}^*} = 2^{n-1} + 1/2$

Proof by application of Yao's Minimax Principle

The upper bound coincides with the general upper bound.

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Remark We already knew this from NFL.

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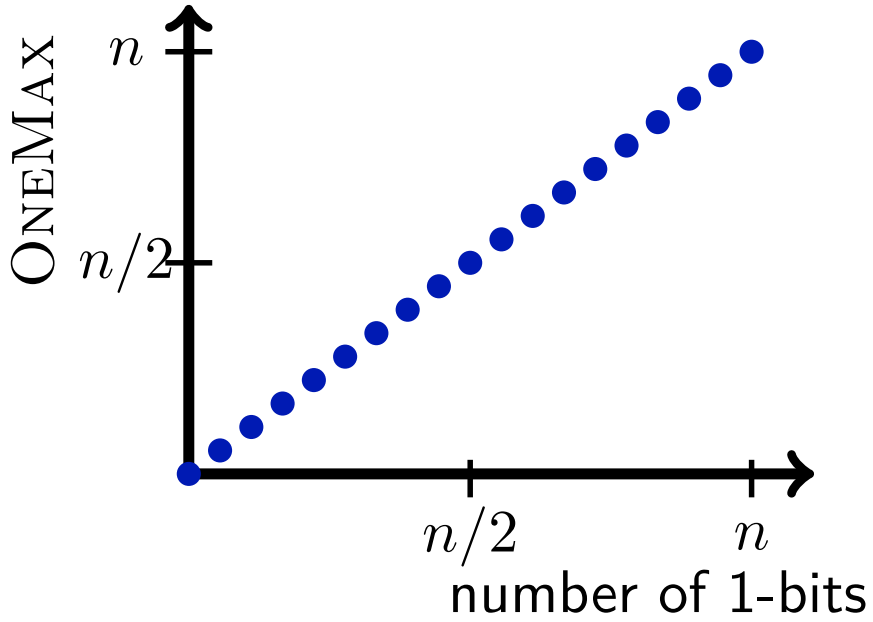

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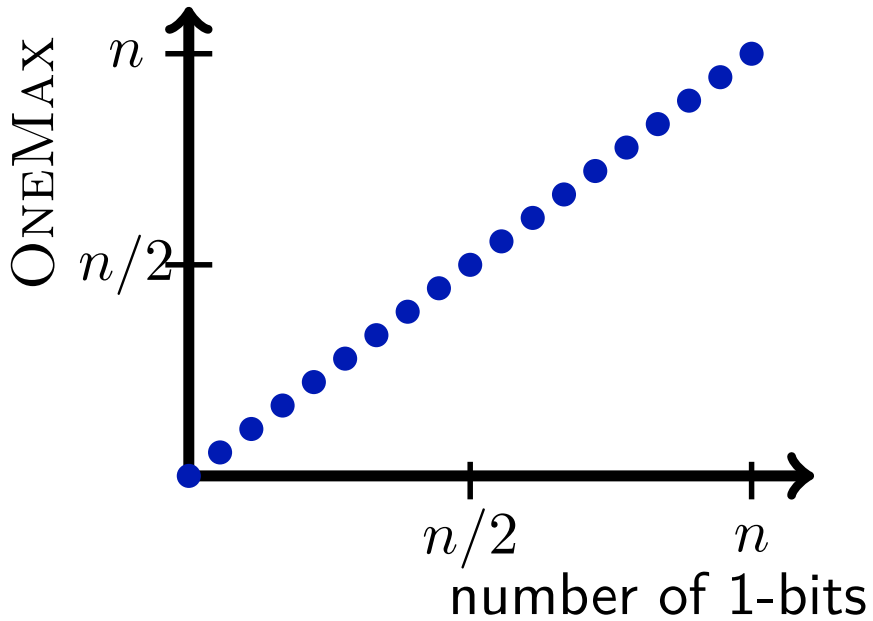

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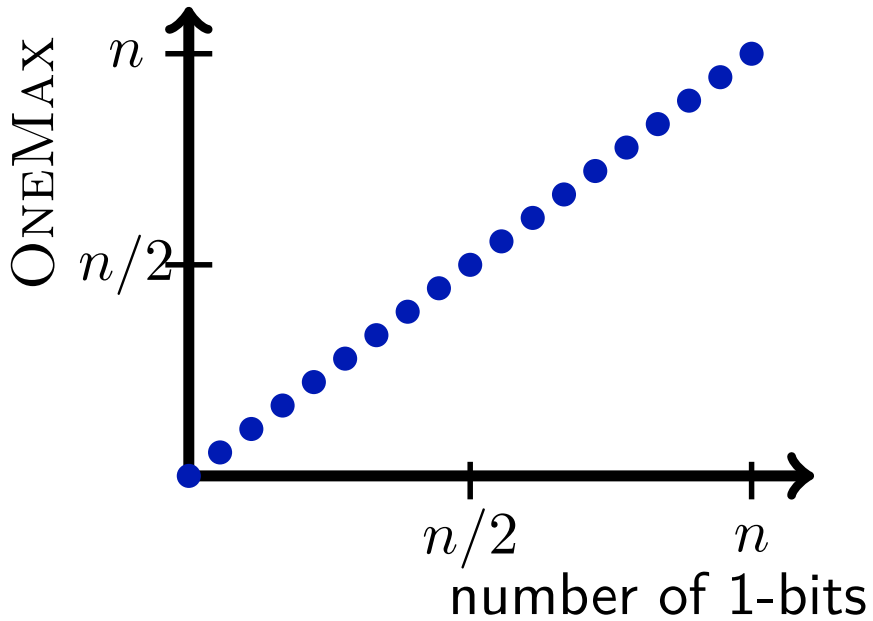

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$\textsc{OneMax}_a(x) = \mathsf{H}(x, \overline{a})$

Hamming distance $\mathsf{H}(x,y) = \sum_{i=1}^{n} x[i] + y[i] - 2x[i]y[i]$

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Remark $B_{\text{OneMax}^*} = O(n)$ is easy to see.

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Does this mean unimodal functions are easy to optimise?

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Remember For any point not optimal under a unimodal function, there exists a path to the global optimum

Unimodal functions

class of unimodal functions

$\mathcal{U} := \{f : \{0,1\}^n \to \mathbb{R} \mid f \text{ unimodal}\}$

What is $B_{\mathcal{U}}$?

We want to find a lower bound on $B_{\mathcal{U}}$.

Remember For any point not optimal under a unimodal function, there exists a path to the global optimum

Definition path of length l is
sequence of l points $p_1, p_2, \ldots, p_l$
with $H(p_i, p_{i+1}) = 1$ for all $1 \leq i < l$

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$$f_P(x) := \begin{cases} n + i & \text{if } x = p_i \text{ and } x \neq p_j \text{ for all } j > i, \\ \textsc{OneMax}(x) & \text{if } x \notin P \end{cases}$$

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$\mathcal{P}_{l(n)} := \{f_P \mid P \text{ has length } l(n)\}$

Random Paths

Construct P with length $l(n)$ randomly:

1. $p_1 := 1^n$; $i := 2$
2. While $i \leq l(n)$ do
3. Choose $p_i \in \{x \mid \mathsf{H}(x, p_{i-1}) = 1\}$ uniformly at random.
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Remark Paths P constructed this way are likely to contain circles.

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For all $f \in \mathcal{U}$ we define

$$\text{Prob}(f) := \begin{cases} p & \text{if } f \in \mathcal{P}_{l(n)} \text{ and } P \text{ is constructed with prob. } p, \\ 0 & \text{otherwise.} \end{cases}$$

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4. giving away for free the knowledge about $p_{l(n)}$ (the global optimum) once p_{j+n} is sampled while p_j is the current known best path point.

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Observation We need a good lower bound on $l'(n)$.

How likely is it to return to old path points?

alternatively What is the probability distribution for the Hamming distance points on the path?

Distance Between Points on the Path

Lemma

$\forall \beta > 0$ *constant*:

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$\forall j \geq \beta n$: $Prob\left(H(p_i, p_{i+j}) \leq \alpha(\beta)n\right) = 2^{-\Omega(n)}$

Summary & Take Home Message

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- Yao's minimax principle

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- Black-box complexity allows for meaningful general lower bounds for RSHs.
- Unimodal problems are not easy to solve.