Coláiste na hOllscoile Corcaigh, Éire
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CS4618 Artificial Intelligence I

Today: Random Processes and Applications

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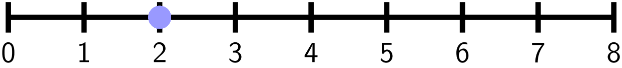
Plans for Today

A Random Process

Consider a random process on $\{0, 1, \ldots, 8\}$



particle moves left with probability p_i^-
particle moves right with probability p_i^+
with $\forall i \in \{0, 1, \ldots, 8\} \colon p_i^- + p_i^+ = 1$

Clearly Can be described by state $X_t \in Z = \{0, 1, \ldots, 8\}$
and transition probabilities $\text{Prob}\left(X_{t+1} = i \mid X_t = j\right)$

Markov Property

$$\text{Prob}\left(X_{t+1} = i \mid X_0 = j_0, X_1 = j_1, \ldots, X_{t-1} = j_{t-1}, X_t = j\right)$$
$$= \text{Prob}\left(X_{t+1} = i \mid X_t = j\right)$$

Markov Chains

Consider Markov chain with state space Z

Definition Markov chain time-homogeneous if $\forall i, j \in Z \colon \forall t, t' \in \mathbb{N}_0 \colon$
$\mathrm{Prob}\left(X_{t+1} = i \mid X_t = j\right) = \mathrm{Prob}\left(X_{t'+1} = i \mid X_{t'} = j\right)$

Notation for time-homogeneous Markov chain
$p_{i,j} = \mathrm{Prob}\left(X_{t+1} = j \mid X_t = i\right)$

$\forall t \in \mathbb{N}_0$ $p_t \in [0;1]^{|Z|}$ describes probability distribution of X_t
$p_t[i] = \mathrm{Prob}\left(X_t = i\right)$

Observation $p_{t+1}[j] = \sum\limits_{i \in Z} p_t[i] \cdot p_{i,j}$

Notation with $P = \left(p_{i,j}\right)$ we have $p_{t+1} = p_t \cdot P$

Observation can be iterated
$\forall t, t' \in \mathbb{N}_0 \colon p_{t+t'} = p_t \cdot P^{t'}$

Properties and Notions

Definition higher order transition probabilities

$$p_{i,j}^{(t)} = \mathrm{Prob}\left(X_t = j \mid X_0 = i\right)$$

Observations

$$p_{i,j}^{(2)} = \sum_k p_{i,k} \cdot p_{k,j}$$

$$p_{i,j}^{(t+1)} = \sum_k p_{i,k}^{(t)} \cdot p_{k,j}$$

$$P^t = \left(p_{i,j}^{(t)}\right)$$

Definition distribution $p \in [0;1]^{|Z|}$ with $p \cdot P = p$ is called stationary distribution

Definition first hitting time of $i \in Z$

$$T_i = \begin{cases} \min\left\{t \mid X_t = i\right\} & \text{if defined} \\ \infty & \text{otherwise} \end{cases}$$

Stopping Times

Observation First Hitting Time T_i
does not depend on future events

Observation reasonable stopping times
should not depend on future events

Definition

A random variable $T\colon \Omega \to \mathbb{N}_0 \cup \{\infty\}$ is a stopping time of a random process $X_0, X_1, X_2, \dots$ if for all $n \in \mathbb{N}_0$ the event $T = n$ can be expressed in terms of $X_0, X_1, \dots, X_n$.

Martingales

Definition

A real-values random process $Y_0, Y_1, Y_2, \ldots$ is called a martingale with respect to a random process $X_0, X_1, X_2, \ldots$ if the following hold for all $n \in \mathbb{N}_0$.

1. Y_n is a function of $X_0, X_1, \ldots, X_n$.
2. $\mathrm{E}\left(|Y_n|\right) < \infty$ or $Y_n \geq 0$
3. $\mathrm{E}\left(Y_{n+1} \mid X_0, X_1, \ldots, X_n\right) = Y_n$

Theorem (Optional Stopping Theorem)

Let $Y_0, Y_1, Y_2, \ldots$ be a martingale with respect to $X_0, X_1, X_2, \ldots$, let T be a stopping time of $X_0, X_1, X_2, \ldots$
If $\exists k \in \mathbb{N}_0 \colon T \leq k$ almost surely or
$T < \infty$ and $\exists k \colon |Y_t| \leq k$ for all $t < T$ almost surely
then $E(Y_T) = E(Y_0)$.

A Fair Random Walk

Consider random process $X_0, X_1, X_2, \ldots$ on $\mathbb{Z}$
with $X_0 = a \in \{0, 1, \ldots, n\}$, $X_{t+1} \in \{X_t + 1, X_t - 1\}$,
$\text{Prob}\left(X_{t+1} = i + 1 \mid X_t = i\right) = \frac{1}{2}$
$\text{Prob}\left(X_{t+1} = i - 1 \mid X_t = i\right) = \frac{1}{2}$

Observation $X_0, X_1, X_2, \ldots$ is a Markov chain

Consider random variable $T := \min\{t \mid |X_t| = n\}$

Observation T is a stopping time for $X_0, X_1, X_2, \ldots$

What is $\text{E}\left(T\right)$?

Expected Duration of the Random Walk

Consider $Y_n := X_n^2 - n$

Claim $Y_0, Y_1, Y_2, \ldots$ is martingale with respect to $X_0, X_1, X_2, \ldots$

$$
\begin{aligned}
& \mathrm{E}\left(Y_{n+1} \mid X_0, X_1, \ldots, X_n\right) \\
= \; & \mathrm{E}\left(X_{n+1}^2 - (n+1) \mid X_0, X_1, \ldots, X_n\right) \\
= \; & \mathrm{E}\left(X_{n+1}^2 - (n+1) \mid X_n\right) = \mathrm{E}\left(X_{n+1}^2 \mid X_n\right) - (n+1) \\
= \; & \frac{1}{2} \cdot (X_n + 1)^2 + \frac{1}{2} \cdot (X_n - 1)^2 - (n+1) \\
= \; & \frac{X_n^2}{2} + X_n + \frac{1}{2} + \frac{X_n^2}{2} - X_n + \frac{1}{2} - n - 1 \\
= \; & X_n^2 - n = Y_n \quad \checkmark
\end{aligned}
$$

Application of the Optional Stopping Theorem

We have $T < \infty$ almost surely

since $\forall t\colon \operatorname{Prob}(T > t) < (1 - 2^{-n})^{\lfloor t/n \rfloor}$

$\Rightarrow$ optional stopping theoreom applicable

$$\operatorname{E}(Y_T) = \operatorname{E}(Y_0) = \operatorname{E}(X_0^2 - 0) = a^2$$

$$\operatorname{E}(Y_T) = \operatorname{E}(X_T^2 - T) = n^2 - \operatorname{E}(T)$$

$$\operatorname{E}(T) = n^2 - \operatorname{E}(Y_T) = n^2 - a^2 = (n - a) \cdot (n + a)$$

End of the Fair Random Walk

We know $\mathrm{E}(T) = n^2 - a^2 = (n-a)\cdot(n+a)$

What is $\mathrm{Prob}(X_T = n)$?

Claim $X_0, X_1, X_2, \ldots$ is martingale with respect to itself

$\mathrm{E}(X_{n+1} \mid X_0, X_1, \ldots, X_n) = \mathrm{E}(X_{n+1} \mid X_n)$

$= \frac{1}{2}\cdot(X_n+1) + \frac{1}{2}\cdot(X_n-1) = X_n$ ✓

Optional Stopping Theorem $\mathrm{E}(X_T) = \mathrm{E}(X_0) = a$

$\mathrm{E}(X_T) = \mathrm{Prob}(X_T = n)\cdot n + \mathrm{Prob}(X_T = -n)\cdot(-n)$

$= \mathrm{Prob}(X_T = n)\cdot n - (1 - \mathrm{Prob}(X_T = n))\cdot n$

$= 2n\mathrm{Prob}(X_T = n) - n$

$\Rightarrow \quad \mathrm{Prob}(X_t = n) = \frac{a+n}{2n} = \frac{1}{2} + \frac{a}{2n}$

The Gambler's Ruin

Consider two gamblers A and B,
initially with s_A€ and s_B€,
playing in rounds, loser of a round pays winner 1€,
independently in each round $\mathrm{Prob}\left(A \text{ wins}\right) = p_A$,
$\mathrm{Prob}\left(B \text{ wins}\right) = p_B$ with $p_A \neq p_B$, $p_A + p_B = 1$

Consider A's funds $X_0 = s_A, X_1, X_2, \ldots$

Observation $X_{t+1} \in \{X_t + 1, X_t - 1\}$ with
$\mathrm{Prob}\left(X_{t+1} = X_t - 1 \mid X_t\right) = p_B$
$\mathrm{Prob}\left(X_{t+1} = X_t + 1 \mid X_t\right) = p_A$

Notation A is ruined if $\exists t\colon X_t = 0$

Questions What is $\mathrm{Prob}\left(A \text{ is ruined}\right)$?
What is the expected duration of the game?

Probability of A's ruin

Theorem

For $p_A \in (0;1)$ *with* $p_A \neq \frac{1}{2}$, *let* $q := \frac{1-p_A}{p_A}$.
$Prob\,(A \text{ is ruined}) = \left(q^{s_A} - q^{s_A+s_B}\right) / \left(1 - q^{s_A+s_B}\right)$

Proof

Observation $X_0, X_1, X_2, \ldots$ is Markov chain

Observation $T := \min\{t \mid X_t \in \{0, s_A + s_B\}\}$
is stopping time for $X_0, X_1, X_2, \ldots$

Observation $T < \infty$ almost surely
since $\forall t\colon \mathrm{Prob}\,(T > t) < \left(1 - p_A^{s_A+s_B}\right)^{\lfloor t/(s_A+s_B)\rfloor}$

A Martingale for A's Ruin

Use $q = \frac{1-p_A}{p_A} = \frac{p_B}{p_A}$

Consider $M_t := q^{X_t}$

Claim M_t is martingale with respect to $X_0, X_1, X_2, \ldots$

$$
\begin{aligned}
& \mathrm{E}\left(M_{t+1} \mid X_0, X_1, \ldots, X_t\right) = \mathrm{E}\left(q^{X_{t+1}} \mid X_t\right) \\
= \; & p_A \cdot q^{X_t+1} + p_B \cdot q^{X_t-1} \\
= \; & q^{X_t} \cdot \left(p_A \cdot q + p_B \cdot \frac{1}{q}\right) \\
= \; & q^{X_t} \cdot (p_B + p_A) = q^{X_t} = M_t \quad ✓
\end{aligned}
$$

Application of Optional Stopping Theorem

We know $M_t = q^{X_t}$ is martingal

Optional Stopping Theorem $\mathrm{E}\left(M_T\right) = \mathrm{E}\left(M_0\right) = q^{s_A}$

$$\mathrm{E}\left(M_T\right) = \mathrm{Prob}\left(A \text{ ruined}\right) \cdot q^0 + \mathrm{Prob}\left(B \text{ ruined}\right) \cdot q^{s_A+s_B}$$
$$= \mathrm{Prob}\left(A \text{ ruined}\right) + \left(1 - \mathrm{Prob}\left(A \text{ ruined}\right)\right) \cdot q^{s_A+s_B}$$
$$= \mathrm{Prob}\left(A \text{ ruined}\right) \cdot \left(1 - q^{s_A+s_B}\right) + q^{s_A+s_B}$$

together $\mathrm{Prob}\left(A \text{ ruined}\right) = \frac{q^{s_A} - q^{s_A+s_B}}{1 - q^{s_A+s_B}}$ □

Observation tells us nothing about duration

Clearly application of optional stopping theorem desirable

Observation we need a different martingale

Expected Duration of the Game

Consider $N_t := X_t - t \cdot (p_A - p_B)$

Claim N_t is martingale with respect to $X_0, X_1, X_2, \ldots$

$$
\begin{aligned}
& \mathrm{E}\left(N_{t+1} \mid N_t = i - t \cdot (p_A - p_B)\right) = \mathrm{E}\left(N_{t+1} \mid X_t = i\right) \\
= \;& p_A \cdot ((i+1) - (t+1) \cdot (p_A - p_B)) \\
& \quad + p_B \cdot ((i-1) - (t+1) \cdot (p_A - p_B)) \\
= \;& (p_A + p_B) \cdot i + p_A - p_B - (p_A + p_B) \cdot (t+1) \cdot (p_A - p_B) \\
= \;& i + p_A - p_B - (t+1) \cdot (p_A - p_B) \\
= \;& i + p_A - p_B - t(p_A - p_B) - p_A + p_B = i - t(p_A - p_B) \\
= \;& N_t \quad ✓
\end{aligned}
$$

Application Optional Stopping Theorem

We have $N_t = X_t - t \cdot (p_A - p_B)$ is martingale

Optional Stopping Theorem $\mathrm{E}(N_T) = \mathrm{E}(N_0) = s_A$

$$\begin{aligned}
\mathrm{E}(N_T) &= \mathrm{Prob}(A \text{ ruined}) \cdot (0 - \mathrm{E}(T) \cdot (p_A - p_B)) \\
&\quad + \mathrm{Prob}(B \text{ ruined}) \cdot ((s_A + s_B) - \mathrm{E}(T) \cdot (p_A - p_B)) \\
&= -\mathrm{E}(T) \cdot \mathrm{Prob}(A \text{ ruined}) \cdot (p_A - p_B) \\
&\quad + (1 - \mathrm{Prob}(A \text{ ruined}))(s_A + s_B) \\
&\quad - \mathrm{E}(T) \cdot (1 - \mathrm{Prob}(A \text{ ruined})) \cdot (p_A - p_B) \\
&= -\mathrm{E}(T) \cdot (p_A - p_B) + (1 - \mathrm{Prob}(A \text{ ruined})) \cdot (s_A + s_B)
\end{aligned}$$

together $\mathrm{E}(T) = \frac{(1 - \mathrm{Prob}(A \text{ ruined}))(s_A + s_B) - s_A}{p_A - p_B}$

Collecting Coupons

Consider Collecting coupons, n different types of coupons

Assumption $\forall i \in \{1, 2, \ldots, n\}$:
each time, independently, $\mathrm{Prob}\,(\text{get coupon type } i) = \frac{1}{n}$

How many coupons do we need until we have a complete collection?

Observation this number is random variable T

Questions What is $\mathrm{E}\,(T)$?
How likely are deviations from $\mathrm{E}\,(T)$?

A Detour: Landau Notation

Observation being less exact can simplify things a lot

Idea Characterise functions according to their "most important term"

Different point of view Characterise functions according to their "order of growth"

Remember for comparing numbers $\leq, \geq, =, <, >$

Define comparisons for functions similarly

Landau Notation

Definition

For $f, g\colon \mathbb{N}_0 \to \mathbb{R}^+$ define

- $f = O(g) \quad :\Leftrightarrow \exists n_0, c > 0\colon \forall n \geq n_0\colon f(n) \leq c \cdot g(n)$
  f grows not faster than g "$\leq$"
- $f = \Omega(g) \quad :\Leftrightarrow g = O(f)$
  f grows not slower than g "$\geq$"
- $f = \Theta(g) \quad :\Leftrightarrow f = O(g) \wedge f = \Omega(g)$
  f and g grow equally fast "$=$"
- $f = o(g) \quad :\Leftrightarrow \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$
  f grows slower than g "$<$"
- $f = \omega(g) \quad :\Leftrightarrow g = o(f)$
  f grows faster than g "$>$"

The Coupon Collector's Problem

Theorem (Coupon Collector's Theorem)

Let T denote the number of coupons obtained until all n types of coupons are present for the first time.

1. $E(T) = n \ln n + O(n)$
2. $\forall \beta > 1\colon \mathit{Prob}\,(T > \beta n \ln n) \leq n^{-(\beta-1)}$
3. $\forall c \in \mathbb{R}\colon \mathit{Prob}\,(T > n \ln n + cn) = 1 - e^{-e^{-c}}$

Summary & Take Home Message

Things to remember

- Markov chains
- martingales
- optional stopping theorem
- gambler's ruin

Take Home Message

- Probability theory is not very complicated.
- Proving useful results is not very difficult.
- Even simple problems can require some amount of analysis.
- Probability theory yields precise and useful answers to practically relevant problems.