Coláiste na hOllscoile Corcaigh, Éire
University College Cork, Ireland

CS4618 Artificial Intelligence I

Today: Random Processes and Applications

Thomas Jansen

October 26th

Plans for Today

A Random Process

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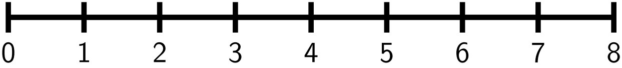

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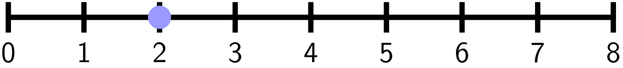

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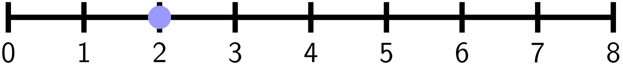

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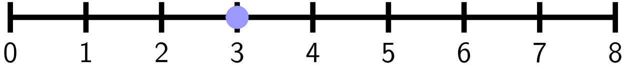

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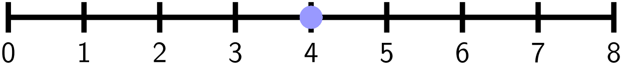

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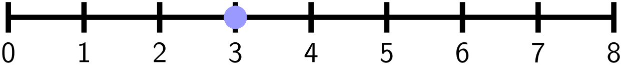

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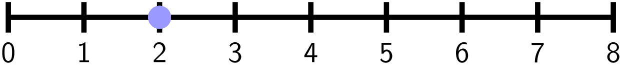

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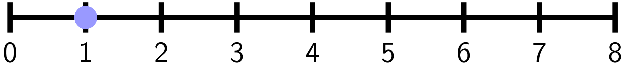

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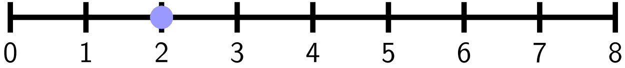

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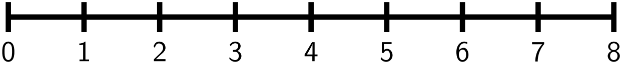

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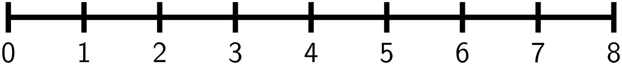

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Markov Property

$$\mathrm{Prob}\left(X_{t+1} = i \mid X_0 = j_0, X_1 = j_1, \ldots, X_{t-1} = j_{t-1}, X_t = j\right)$$
$$= \mathrm{Prob}\left(X_{t+1} = i \mid X_t = j\right)$$

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Observation can be iterated
$\forall t, t' \in \mathbb{N}_0\colon p_{t+t'} = p_t \cdot P^{t'}$

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Definition first hitting time of $i \in Z$

$$T_i = \begin{cases} \min\{t \mid X_t = i\} & \text{if defined} \\ \infty & \text{otherwise} \end{cases}$$

Stopping Times

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Definition

A random variable $T\colon \Omega \to \mathbb{N}_0 \cup \{\infty\}$ is a stopping time of a random process $X_0, X_1, X_2, \ldots$ if for all $n \in \mathbb{N}_0$ the event $T = n$ can be expressed in terms of $X_0, X_1, \ldots, X_n$.

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A real-values random process $Y_0, Y_1, Y_2, \ldots$ is called a martingale with respect to a random process $X_0, X_1, X_2, \ldots$ if the following hold for all $n \in \mathbb{N}_0$.

1. Y_n is a function of $X_0, X_1, \ldots, X_n$.
2. $\mathrm{E}\left(|Y_n|\right) < \infty$ or $Y_n \geq 0$
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Theorem (Optional Stopping Theorem)

Let $Y_0, Y_1, Y_2, \ldots$ be a martingale with respect to $X_0, X_1, X_2, \ldots$, let T be a stopping time of $X_0, X_1, X_2, \ldots$
If $\exists k \in \mathbb{N}_0 \colon T \leq k$ almost surely or
$T < \infty$ and $\exists k \colon |Y_t| \leq k$ for all $t < T$ almost surely
then $E(Y_T) = E(Y_0)$.

A Fair Random Walk

Consider random process $X_0, X_1, X_2, \ldots$ on $\mathbb{Z}$
with $X_0 = a \in \{0, 1, \ldots, n\}$, $X_{t+1} \in \{X_t + 1, X_t - 1\}$,
$\mathrm{Prob}\left(X_{t+1} = i + 1 \mid X_t = i\right) = \frac{1}{2}$
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What is $\mathrm{E}\left(T\right)$?

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End of the Fair Random Walk

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$\Rightarrow \quad \mathrm{Prob}(X_t = n) = \frac{a+n}{2n} = \frac{1}{2} + \frac{a}{2n}$

The Gambler's Ruin

Consider

two gamblers A and B,
initially with s_A€ and s_B€,
playing in rounds, loser of a round pays winner 1€,
independently in each round $\mathrm{Prob}\,(A \text{ wins}) = p_A$,
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Consider A's funds $X_0 = s_A, X_1, X_2, \ldots$

Observation $X_{t+1} \in \{X_t + 1, X_t - 1\}$ with
$\text{Prob}\,(X_{t+1} = X_t - 1 \mid X_t) = p_B$
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Questions What is $\mathrm{Prob}\,(A \text{ is ruined})$?
What is the expected duration of the game?

Probability of A's ruin

Theorem

For $p_A \in (0;1)$ *with* $p_A \neq \frac{1}{2}$, *let* $q := \frac{1-p_A}{p_A}$.
$Prob\,(A \text{ is ruined}) = \left(q^{s_A} - q^{s_A+s_B}\right) / \left(1 - q^{s_A+s_B}\right)$

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A Martingale for A's Ruin

Use $q = \frac{1-p_A}{p_A} = \frac{p_B}{p_A}$

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$$\mathrm{E}\left(M_{t+1} \mid X_0, X_1, \ldots, X_t\right)$$

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✔

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We know $M_t = q^{X_t}$ is martingal

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Observation tells us nothing about duration

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Observation we need a different martingale

Expected Duration of the Game

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Consider $N_t := X_t - t \cdot (p_A - p_B)$

Claim N_t is martingale with respect to $X_0, X_1, X_2, \ldots$

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Application Optional Stopping Theorem

We have $N_t = X_t - t \cdot (p_A - p_B)$ is martingale

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together $\mathrm{E}\,(T) = \frac{(1 - \mathrm{Prob}(A \text{ ruined}))(s_A + s_B) - s_A}{p_A - p_B}$

Collecting Coupons

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Questions What is $\mathrm{E}\left(T\right)$?
How likely are deviations from $\mathrm{E}\left(T\right)$?

A Detour: Landau Notation

Observation being less exact can simplify things a lot

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Remember for comparing numbers $\leq, \geq, =, <, >$

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Different point of view Characterise functions according to their "order of growth"

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Define comparisons for functions similarly

Landau Notation

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For $f, g \colon \mathbb{N}_0 \to \mathbb{R}^+$ define

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The Coupon Collector's Problem

Theorem (Coupon Collector's Theorem)

Let T denote the number of coupons obtained until all n types of coupons are present for the first time.

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3. $\forall c \in \mathbb{R}\colon \mathit{Prob}\,(T > n \ln n + cn) = 1 - e^{-e^{-c}}$

Summary & Take Home Message

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- Probability theory is not very complicated.

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- Proving useful results is not very difficult.
- Even simple problems can require some amount of analysis.
- Probability theory yields precise and useful answers to practically relevant problems.