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# CS4618 Artificial Intelligence I

## Today: Informed Search

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# Plans for Today

# Informed Search

**Remember** uninformed search ($\widehat{=}$ blind search)

- problem given by
  - initial state
  - possible actions
  - transition model
  - goal test
  - path cost
- blind search $\widehat{=}$ search algorithm knows only problem

**Informed Search** additionally evaluation function $f$ used

$f(v)$ estimates cost of node $v$

Application algorithm explores node $v$ with $f(v) = \min\{f(w) \mid w \in \text{frontier}\}$

$\widehat{=}$ best-first search (like uni.-cost search)

often $f$ incorporates heuristic component

heuristic component $\widehat{=}$ additional knowledge about problem

# Heuristic Functions

**Remember** algorithm explores node $v$
with $f(v) = \min\{f(w) \mid w \in \text{frontier}\}$
($\widehat{=}$ best-first search)
$f$ incorporates heuristic function $h$

**Definition** heuristic function $h$
maps states to estimated cost of a cheapest path (of actions) from the state to a goal state

**Consequences**

- $h(v) = 0 \Leftrightarrow v$ is goal state
- $h(v) \geq 0$ (since we assume non-negative cost)

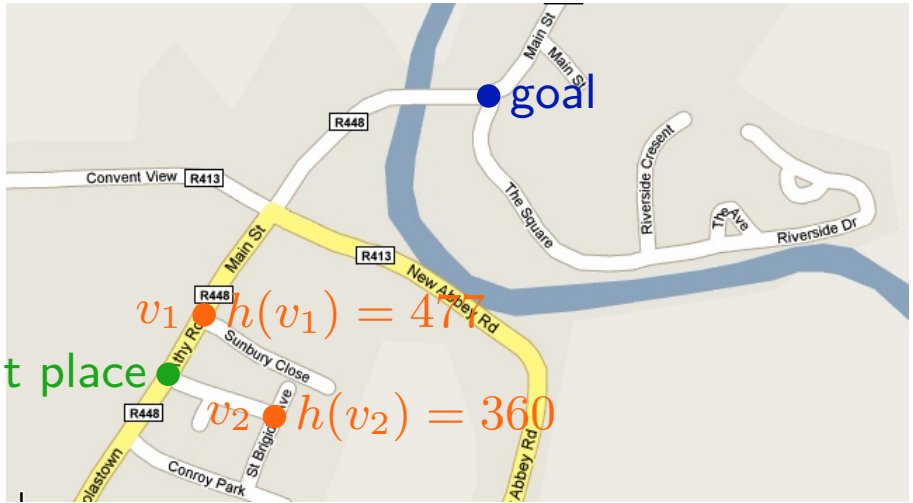
**Remark** $h$ depends on states
not on the path to the state in the search tree

# Greedy Best-First Search

**Remember** informed search algorithm explores node $v$ with $f(v) = \min\{f(w) \mid w \in \text{frontier}\}$ ($\widehat{=}$ best-first search)
$f$ incorporates heuristic function $h$

**Definition** of greedy best-first search $f = h$

**Example** travelling from Cork to Dublin by bike
goal minimise actual travel distance
possible additional heuristic information
$\forall$places: straight-line distance place$\leftrightarrow$Dublin




Observe may lead to wrong decisions and even infinite loops

# Properties of Greedy Best-First Search

- Completeness
  in general no
  but yes if state space finite and no nodes are 'repeated'
- Optimality
  in general no
- Time Complexity
  in 'tree version' $O(b^m)$
- Space Complexity
  in 'tree version' $O(b^m)$

Why consider greedy best-first search at all?

Fact heuristic function $h$ matters
may lead to substantial reduction in time and space complexity

# A* Search

**Remember** informed search algorithm explores node $v$
with $f(v) = \min\{f(w) \mid w \in \text{frontier}\}$
($\widehat{=}$ best-first search)
$f$ incorporates heuristic function $h$

**Definition** of A* search $f = g + h$
$g(v) =$ path cost from start node to $v$
(Remark slight abuse of notation due to different domains)

**Remember** heuristic function $h$
maps states to estimated cost of a cheapest path
(of actions) from the state to a goal state

**Thus** $f(v) =$ estimated cost of cheapest solution via $v$

**Remark** depending on $h$ A* search can be complete and optimal

# Admissible Heuristics

## Definition (Admissibility)

A heuristic function $h$ is called admissible if and only if

$$\forall s \in \text{state space}: h(s) \leq \min \{\text{cost}(a_1, a_2, \ldots, a_n) \mid \text{transition-model}(\text{transition-model}(\cdots (\text{transition-model}(s, a_1), a_2), \ldots, a_n) \text{ is goal}\}$$

holds.

### Observation

- admissible $h$ never overestimates
- admissible $h$ are optimistic

**Example** straight-line distance for street distances

# Consistent Heuristics

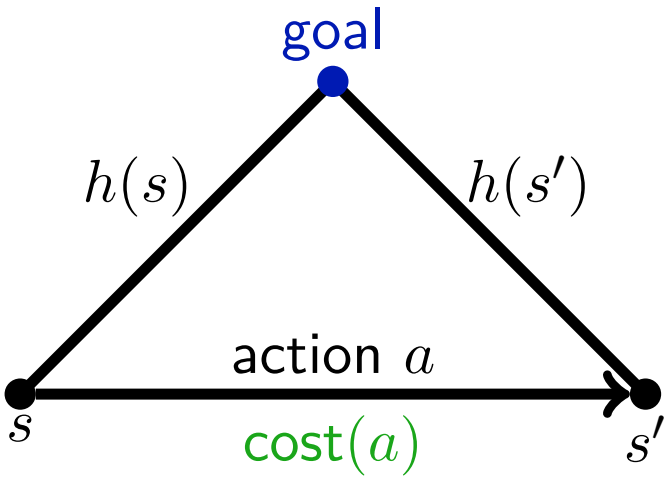
## Definition (Consistency)

A heuristic function $h$ is called consistent if and only if

$$\forall s \in \text{state space}, \text{applicable action } a,$$
$$s' = \text{transition-model}(s, a) \colon h(s) \leq \text{cost}(a) + h(s')$$

holds.

Remark sometimes also called monotone




Observation monotonicity $\widehat{=}$ triangle inequality

Observation monotonicity $\Rightarrow$ admissibility

# Admissibility and Consistency

**Remember** $h$ admissible

$\Leftrightarrow \forall s \in$ state space: $h(s) \leq \min \{\text{cost}(a_1, a_2, \ldots, a_n) \mid$ transition-model(transition-model($\cdots$ (transition-model$(s, a_1), a_2), \ldots, a_n)$ is goal$\}$

(no overestimating)

$h$ consistent

$\Leftrightarrow \forall s \in$ state space, applicable action $a$, $s' =$ transition-model$(s, a)$: $h(s) \leq \text{cost}(a) + h(s')$

(triangle inequality)

**Claim** $h$ consistent $\Rightarrow$ $h$ admissible

**Consider** minimal distance $d$ from $s$ to goal

**Observe** $d = 0$ no action, $h(s) = 0$ ✓

**Observe** $d = 1$ $h(s) \leq \text{cost}(a) + h(s') = \text{cost}(a)$ ✓

**Observe** $d > 1$ consider one step and $d - 1$ by induction ✓

# A* Search with Consistent Heuristics

**Claim** A* search with a consistent $h$ is optimal

**Remember** $\underbrace{f}_{\text{evaluation}} = \underbrace{g}_{\text{cost}} + \underbrace{h}_{\text{heuristic}}$

**Sub-Claim 1** along any path $f$-values are non-decreasing

**Proof**

**Consider** state $s$, action $a$, state $s' = \text{transition-model}(s, a)$

$$f(s') \underbrace{=}_{\text{definition}} g(s') + h(s') \underbrace{=}_{\text{by transition}} g(s) + \text{cost}(a) + h(s')$$

$$\underbrace{\geq}_{\text{consistency}} g(s) + h(s) \underbrace{=}_{\text{definition}} f(s) \checkmark$$

for each edge, thus for each path □

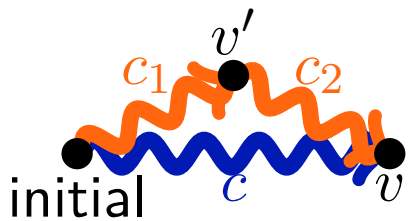
# A* Search with Consistent Heuristics (continued)

**Claim** A* search with a consistent $h$ is optimal

**Sub-Claim 1** along any path $f$-values are non-decreasing ✓

**Sub-Claim 2** A* selects $v$ for expansion
$\Rightarrow$ optimal path to $v$ found

**Proof**
**Assume** otherwise $\exists$




with $c > c_1 + c_2$

Observe $f(v) \geq f(v')$ (due to Sub-Claim 1)

Thus $v'$ expanded prior to $v$

Thus optimal route found prior to expanding $v$ ↯ □

# A* Search with Consistent Heuristics (continued)

**Claim** A* search with a consistent $h$ is optimal

**Sub-Claim 1** along any path $f$-values are non-decreasing ✓

**Sub-Claim 2** A* selects $v$ for expansion

$\Rightarrow$ optimal path to $v$ found ✓

**Observation** A* expands nodes in non-decreasing $f$-order
follows from Sub-Claim 1 and Sub-Claim 2

**Consider** first considered goal node $v$

**Observation** $f(v) = g(v)$ (true cost)
since $h(v) = 0$ for all goal nodes (by definition)

**Consequence**

- first considered goal node has minimal cost ✓
- A* is optimal ✓

# A* Search and Uniform-Cost Search

**Remember** uniform-cost search
use priority queue and sort by cost $g$
$\widehat{=} f = g$

**Remember** A* search $\widehat{=} f = g + h$

**Observation** $(f = g) \equiv (f = g + 0)$
thus uniform-cost search $\widehat{=}$ A* search with $h = 0$
(A* search without heuristic information)
'without heuristic information'
$\widehat{=}$ 'with trivial heuristic information 0'

**Remember** uniform-cost search is optimal
A* with consistent heuristic is optimal

Is $0$ consistent heuristic information?

# Consistency of 0 Information

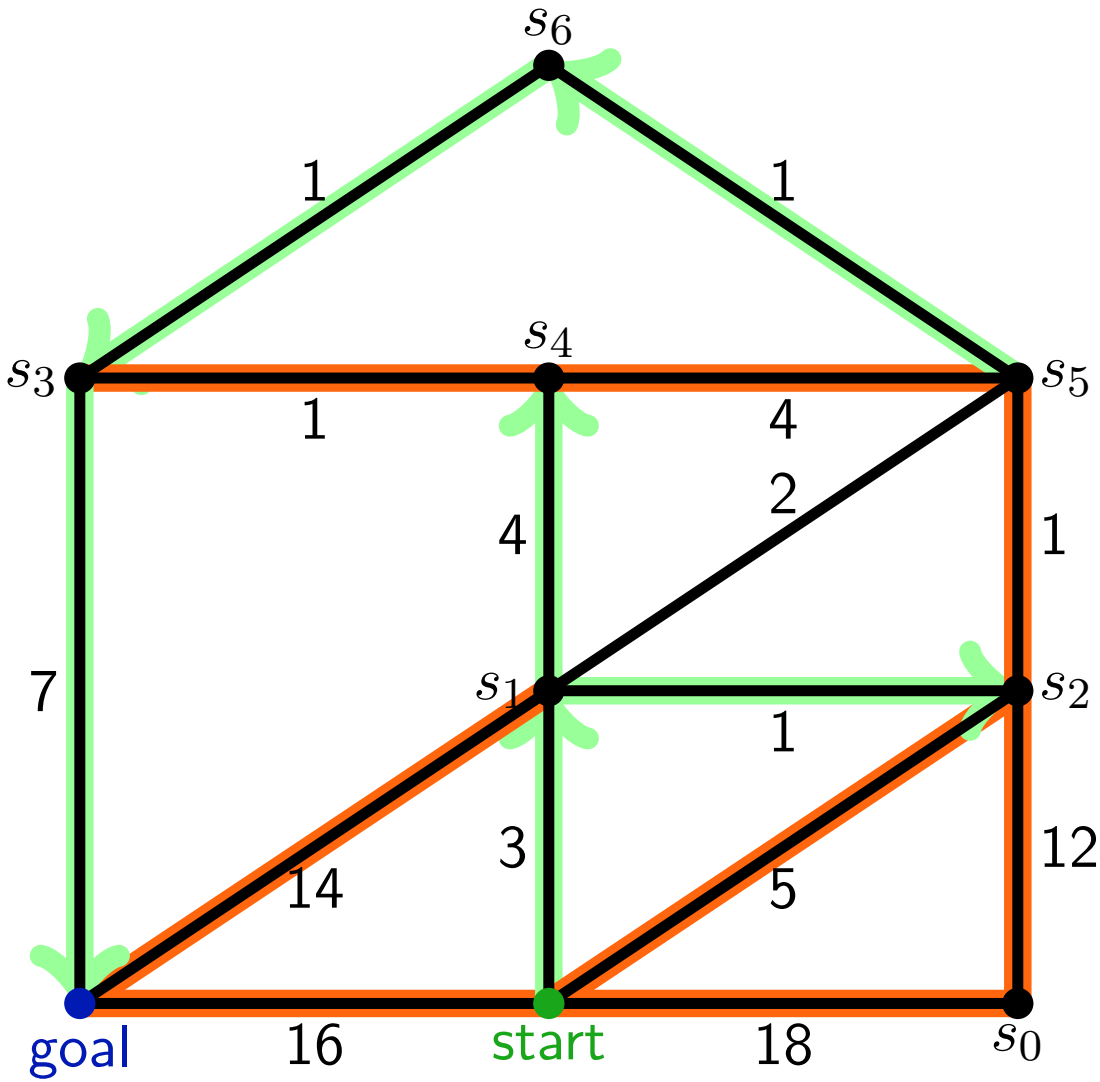
**Remember** uniform-cost search is optimal
A* with consistent heuristic is optimal

Is $0$ consistent heuristic information?

**Remember** $h$ consistent
$\Leftrightarrow \forall s \in$ state space, applicable action $a$,
$s' =$ transition-model$(s, a)$: $h(s) \leq \text{cost}(a) + h(s')$
(triangle inequality)

**Observation** with $h = 0$ we have $\forall v$: $h(v) = 0$
thus We need $0 \leq \text{cost}(a) + 0$
holds since we assume non-negative cost,
$\forall a$: $\text{cost}(a) \geq 0$ ✓

# Example 'Shortest Path' Without Heuristic Information




frontier

$s_0$

explored

start, $s_1$, $s_2$, $s_5$, $s_6$, $s_3$, $s_4$,

goal

# Contours in State Space

**Consider** metric in state space
given by actual minimum path cost

**Observation** A* search with $h = 0$ ($\widehat{=}$ uniform-cost search)
explores state space in circles around initial state

**Observation** A* search with $h \neq 0$
explores state space according to $g + h$
$\rightsquigarrow$ circles 'bend' towards promising regions
(promising according to $h$)

**Observation** A* search implements biased search
with search bias defined by $h$

# Properties of A$^*$ Search

**Consider** A$^*$ search with consistent $h$
in problem with optimal solution with cost $c_{\text{opt}}$

- A$^*$ is complete
  if there are finitely many states with cost $\leq c_{\text{opt}}$
- A$^*$ is optimal
  if it is complete
- A$^*$ prunes parts of the search space
  it never expands nodes with cost $> c_{\text{opt}}$
- A$^*$ has optimal efficiency
  no complete, optimal search algorithm can guarantee to explore less nodes
- $A^*$ is often too inefficient to solve problems in practice
- $A^*$ encounters space issues before time issues
  since it needs to store all expanded nodes

# Memory-Bounded Heuristic Search

Aim at solving the space complexity issue of A* search

- iterative-deepening A* search (IDA*)
  use $f = g + v$-value as cutoff measure (not depth): cutoff is min. $f$-value of a node exceeding the previous cutoff value
  problematic for real-valued functions $f$
  re-considers large parts of the search space many times
- recursive best-first search (RBFS)
  restrict best-first search by value of best-so-far alternative solution, backtrack if limit reached, keep track of branches with potential solutions
  still re-considers large parts of the search space many times
- simplified memory-bounded A* search (SMA*)
  work like A* until memory-limit is reached, then discard worst leaf memorising its value, re-evaluate discarded sub-trees if all other sub-trees look worse
  re-consideration can increase time complexity drastically

# On Heuristic Functions

**Remember** performance of A* search with (consistent) heuristic $h$ depends very much on $h$

Can we characterise the influence of $h$?

**Remember** example 8-puzzle

start state

| | | |
|---|---|---|
| 7 | 2 | 4 |
| 5 | | 6 |
| 8 | 3 | 1 |

goal state

| | | |
|---|---|---|
| | 1 | 2 |
| 3 | 4 | 5 |
| 6 | 7 | 8 |

transitions, e. g.,

| | | |
|---|---|---|
| 7 | 2 | 4 |
| 5 | | 6 |
| 8 | 3 | 1 |

$\xrightarrow{\text{up}}$

| | | |
|---|---|---|
| 7 | 2 | 4 |
| 5 | 3 | 6 |
| 8 | | 1 |

# On 8-Puzzle

## General Properties

- number of different states $\leq 9! = 362\,880$
  manageable for any non-repeating search algorithm
- average branching factor $= 1 \cdot 4 + 4 \cdot 3 + 4 \cdot 2 = \frac{24}{9} \approx 2.7$
- comparable to 15-puzzle (well studied, interesting)

**Idea** empirically compare effects of different heuristics on A*'s search behaviour

**Need** at least two different heuristic functions $h_1$, $h_2$ perferably consistent

# Heuristics for A* Search

- $h_1(v) = \#\text{misplaced tiles}$
  Admissible?
  Remember yes, if never overestimating
  yes, since per misplaced tile at least one move necessary ✓
  Consistent?
  Remember yes, if $h(s) \leq h(\text{transition-model}(s, a)) + \text{cost}(a)$
  yes, since each move changes number of misplaced tiles by $\leq 1$ and $\forall a\colon \text{cost}(a) = 1$ ✓
- $h_2(v) = \sum_{i=1}^{8} \text{Manhattan distance}(\text{tile } i, \text{goal position})$
  Admissible? yes, since each move decreases Manhattan distance by at most 1 ✓
  Consistent? yes, since each move changes Manhattan distance by $\leq 1$ and $\forall a\colon \text{cost}(a) = 1$ ✓

# Applying Heuristics $h_1$ and $h_2$

## Remember

- $h_1(v) = \#\text{misplaced tiles}$
- $h_2(v) = \sum_{i=1}^{8} \text{Manhattan distance}(\text{tile } i, \text{goal position})$

## Example

- $h_1 \left( \begin{array}{|c|c|c|} \hline 7 & 2 & 4 \\ \hline 5 & & 6 \\ \hline 8 & 3 & 1 \\ \hline \end{array} \right) = 8$
- $h_2 \left( \begin{array}{|c|c|c|} \hline 7 & 2 & 4 \\ \hline 5 & & 6 \\ \hline 8 & 3 & 1 \\ \hline \end{array} \right) = 3 + 1 + 2 + 2 + 2 + 3 + 3 + 2 = 18$
- Info true cost $\left( \begin{array}{|c|c|c|} \hline 7 & 2 & 4 \\ \hline 5 & & 6 \\ \hline 8 & 3 & 1 \\ \hline \end{array} \right) = 26$

# Evaluating Heuristics for 8-Puzzle

## What do we compare?

- number of generated nodes (for solution at given depth)
- effective branching factor (for solution at given depth)
  for a uniform search tree with $n+1$ nodes with branching factor $b$ we have

$$n+1 = \sum_{i=0}^{d} b^i = \left(b^{d+1} - 1\right)/(b-1)$$

| | number of generated nodes | | | effective branching factor | | |
|---|---|---|---|---|---|---|
| $d$ | IDS | A*$(h_1)$ | A*$(h_2)$ | IDS | A*$(h_1)$ | A*$(h_2)$ |
| 2 | 10 | 6 | 6 | 2.45 | 1.79 | 1.79 |
| 4 | 112 | 13 | 12 | 2.87 | 1.48 | 1.45 |
| 6 | 680 | 20 | 18 | 2.73 | 1.34 | 1.30 |
| 8 | 6 384 | 39 | 25 | 2.80 | 1.33 | 1.24 |
| 10 | 47 127 | 93 | 39 | 2.79 | 1.38 | 1.22 |
| 12 | 3 644 035 | 227 | 73 | 2.78 | 1.42 | 1.24 |

Observation $h_2$ dominates $h_1$

# Domination of Heuristics

**Remember** $h_2$ dominates $h_1$

$h_1(v) = \#\text{misplaced tiles}$

$h_2(v) = \sum_{i=1}^{8} \text{Manhattan distance}(\text{tile } i, \text{goal position})$

**Remember** A* never expands nodes $v$ with $f(v) > c_{\text{opt}}$

A* expands each node $v$ with $f(v) < c_{\text{opt}}$

A* expands each node $v$ with $g(v) + h(v) < c_{\text{opt}}$

A* expands each node $v$ with $h(v) < c_{\text{opt}} - g(v)$

**Observation** $\forall v\colon h_2(v) \geq h_1(v)$

**Consequence** A*$(h_2)$ will not explore more nodes than A*$(h_1)$
(and most likely much less)
$\widehat{=}$ $h_2$ dominates $h_1$

# Summary & Take Home Message

## Things to remember

- informed search: evaluation f. $f$, path cost $g$, heuristic f. $h$
- greedy best-first search ($f = h$), A* search ($f = g + h$)
- heuristic functions: admissibility (no overestimation), consistency (triangle inequality)
- A* variants: iterative-deepening A* (IDA*), recursive best-first (RBFS), simplified memory-bounded A* (SMA*)
- heuristics for A*: domination

## Take Home Message

- Informed search can hugely excel over uninformed search.
- A* search is very powerful but not always practical.
- Variants of A* search tackling some of A*'s problems exist.
- Heuristics are crucial for the performance of A* search.