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Bounds for isoperimetric constants of infinite plane graphs

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Abstract

We establish an upper bound and new lower bounds for isoperimetric constants over infinite locally finite plane graphs with one end and with given minimum degree $\delta \geq 7$. © 2001 Elsevier Science B.V. All rights reserved.

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The subject matter of this note is hyperbolic infinite triangulations, and, generally, infinite cell 2-complexes G with *one end* – that is, the carrier of G is homeomorphic to the plane or, in other words, all points in the plane are covered by G . We draw parallels between the metric properties of the cell 2-complexes and their continuous counterparts. This approach gives rise to several discrete extremal problems.

Let M^2 be a topological 2-manifold. Let G be a cell 2-complex homeomorphic to M^2 (more accurately, the carrier of G is homeomorphic to M^2). Especially, G is a *triangulation* of M^2 when it is simplicial. It is a well-known fact that if M^2 is noncompact (compact, respectively) then G is infinite (finite). By $V(G)$, $E(G)$, and $F(G)$ we denote the sets of the vertices (0-cells), edges (1-cells), and faces (2-cells) of G , respectively. The *degree* of $v \in V(G)$ is the number of the edges of which v is a vertex.

The following is a classical geometric fact which provides background and motivation for the present research. Let M^2 be a simply connected Riemannian 2-manifold

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which is complete as a metric space. If the Gaussian curvature K is uniformly bounded above by a negative constant,

$$K(p) \leq -\varepsilon^2 < 0, \quad (1)$$

for each point $p \in M^2$, then for any piecewise smooth closed curve C bounding a 2-disk D ,

$$\frac{\text{length } C}{\text{area } D} \geq \varepsilon. \quad (2)$$

Letting C vary and taking the infimum of these ratios leads to a positive number which is an important type of the so-called isoperimetric constant of M^2 (see [2,5,14]), which we will refer to as the Cheeger–Gromov–Yau constant.

Let G be a cell 2-complex homeomorphic to a topological 2-manifold M^2 with empty boundary. For a vertex $v \in V(G)$, the *combinatorial curvature* $K_c(v)$ is defined to be 2π minus the sum of the angles around v , where each face of G is thought of as a Euclidean regular polygon. Thus, G models a metric of M^2 by way of the combinatorial curvature K_c which is a discrete analogue of the Gaussian curvature K . In this interpretation, if G is δ -regular, restriction on the minimum degree δ of G becomes a restriction on the curvature of M^2 . The larger δ , the more curved (negatively) M^2 is. The hyperbolicity of M^2 is guaranteed by the condition $\delta \geq 7$. This condition is a discrete analogue of condition (1), while a 6-regular triangulation models a flat 2-manifold.

We deal with a specific situation in which M^2 is the plane and G is a *plane graph* – that is, an embedding of a planar locally finite graph (1-complex) in the plane. Thus, a plane graph is a two-dimensional object; it is determined by the triple of sets $\{V(G), E(G), F(G)\}$. We restrict our attention to the case in which G has one end. This restriction, along with the condition $\delta \geq 7$ (or even $\delta \geq 6$), guarantees that G corresponds to a *cell 2-complex* in which every face is bounded by a cycle (no repeated vertices), and two faces may intersect only at a single vertex or a single edge. A *plane cell 2-complex* is one homeomorphic to the plane. It is necessarily infinite and locally finite [8]. The case of a single end is of particular interest, thanks to the fact that some infinite locally finite plane graphs with one end are the universal coverings of finite graphs on closed compact surfaces. Furthermore, it has turned out [12] that such plane graphs also have a structure similar to that of graph embeddings on surfaces with large representativity and minimum degree of at least six.

A *finite plane subgraph* of G is a plane embedding of the graph determined by fixing finite subsets of $V(G)$ and $E(G)$, naturally inherited from G . Generally, a plane subgraph is not required to be a subcomplex of G .

Let H be a finite plane subgraph of G . Denote by $a(H)$ the number of finite faces of H (the “area” of H), and by $l(\partial H)$ the number of the edges of H that are on the boundary of at most one finite face (the “perimeter” of H). By analogy with the continuous case, define the *discrete isoperimetric constant* $h(G)$ by

$$h(G) = \inf_H \frac{l(\partial H)}{a(H)}, \quad (3)$$

and define the δ -isoperimetric constant $h(\delta)$ by

$$h(\delta) = \inf_G h(G), \quad (4)$$

where the infimum is taken over all plane cell 2-complexes G with minimum degree δ .

We show that we may assume that G is a triangulation when estimating (4) and that H is a *polygon* in G when estimating (3). By a polygon we mean a subcomplex of G , the carrier of which is homeomorphic to a closed 2-disk. The boundary of H is denoted by ∂H .

We estimate $h(G)$ and $h(\delta)$ for plane cell 2-complexes G with $\delta \geq 7$. It can be easily seen that $h(G) = 0$ if G is a 6-regular plane triangulation. A number of people (Dodziuk, Mohar et al. [4,1,7,9–11]) have looked at the isoperimetric constant $h(G)$, sometimes in the form of the plane dual. Basically, only two bounds on the discrete isoperimetric constants were known previously, both obtained by combinatorial methods. Here for uniformity we restate those results in terms of $h(\delta)$. The first is a constant bound obtained by Dodziuk [4]:

$$h(\delta) \geq \frac{1}{26} \quad (\delta \geq 7). \quad (5)$$

Historically, this bound appears to be the first discrete analogue of inequality (2). (A similar inequality also appears, in another context, in DeBaun [3].) The Dodziuk bound (5) coupled with the uniform boundedness of the degree of a vertex from above ensures the transience of random walks on G . This means that a particle starting from a vertex v escapes to infinity with probability one, for any vertex $v \in V(G)$ (see [4]). Bound (5) was significantly improved by Mohar [10], who showed that

$$h(\delta) \geq \frac{\delta - 6}{\delta - 4} \quad (\delta \geq 7). \quad (6)$$

The primary aim of this note is to derive a still better lower bound:

$$h(\delta) \geq \frac{(\delta - 6)(\delta^2 - 8\delta + 15)}{(\delta - 4)(\delta^2 - 8\delta + 13)} \quad (\delta \geq 7). \quad (7)$$

Our proof is combinatorial, and here is a sketch: We show that the bound in (7) is a lower bound for $h(G)$ for any plane cell 2-complex G with minimum degree δ . Our lens structure theorem says that a plane cell 2-complex G with $\delta \geq 7$ (or even with $\delta \geq 6$) has a lens structure – that is, G is the limit of the *lens sequence* $\{L_n\}_{n=0}^\infty$ centered at a fixed vertex $v_0 \in V(G)$, defined by recursion on n as follows: $L_0 = \{v_0\}$, and, for $n \geq 1$, L_n is obtained from L_{n-1} by adjoining the faces of G incident to ∂L_{n-1} that do not lie in L_{n-1} . Clearly, each L_n is a polygon. A vertex on ∂L_n is said to be a Type I or Type II vertex according to whether it has, respectively, one or two neighbors on ∂L_{n-1} . The contribution of a vertex to the growth of the graph (as $n \rightarrow \infty$) essentially depends on its type. Denote by A_n the triangulated annulus bounded by ∂L_n and ∂L_{n+1} . Given a polygon Q in G , we represent $Q \cap A_n$ as the union of sections of certain Types where the Type of a single section R is determined by the Type of one of the

endvertices of the path $R \cap \partial L_n$. We estimate how much a single section can affect the ratio in (3), and, finally, show that the bound in (7) is a lower bound for $h(G)$.

It is worth mentioning that Mohar [10] studied infinite triangulations in a more general setting: he triangulated open subsets of surfaces with non-negative Euler characteristic, possibly with multiple ends. Most of his bounds are tight. However, in the case of infinite plane graphs with one end, his bound (6) is not tight and we improve his bound to that given in (7). The difference between the upper bound (9) and our bound (7) is $O(\delta^{-6})$, while the difference between (9) and the Mohar bound (6) is $O(\delta^{-2})$ (as $\delta \rightarrow \infty$).

Here we also include a lower bound for $h(G)$ where G is a δ -regular triangulation of the plane with $\delta \geq 7$. This bound is geometric; it is immediately derived from (2) by a geometric argument, although is far from being tight. Anyhow, the method we use to obtain it demonstrates that the isoperimetric constant (3) is indeed an analogue of the Cheeger–Gromov–Yau constant. We exploit the idea of Grünbaum and Shephard [6] using the Poincaré model of the hyperbolic plane H^2 with constant negative curvature $K = -1$. Draw an equilateral triangle Δ with angles of size $\alpha = 2\pi/\delta$, which has one of its vertices, v_0 , in the center of the Poincaré disk. Then tile H^2 by repeatedly reflecting that triangle about its sides. The infinite tessellation of H^2 so-obtained is a geometric realization of G . (From this construction it follows that G has a lens structure, where each lens L_n may be thought of as representing one side of an optic hyperbolic lens.) Remarkably, given a polygon Q in G , the combinatorial quantities $l(\partial Q) = |E(\partial Q)|$ and $a(Q) = |F(Q)|$ agree with the perimeter and area of Q , respectively. We need only to multiply $l(\partial Q)$ by $L(\Delta)$, the length of the side of Δ , to get the perimeter of Q and multiply $a(Q)$ by $A(\Delta)$, the area of Δ , to get the area of Q . Then, applying inequality (2) to the piecewise smooth curve $\hat{\partial}Q$ leads to

$$h(G) \geq \frac{A(\Delta)}{L(\Delta)} = \frac{\pi(\delta - 6)}{2\delta} \frac{1}{\ln(D + \sqrt{D^2 - 1})} \quad (\delta \geq 7), \quad (8)$$

where $D = (2 \sin \pi/\delta)^{-1}$. We omit the details and refer the interested reader to Ratcliffe [13] for foundations of hyperbolic geometry. Although bound (8) follows from known geometric results, we were unable to find any account of it in the literature. Interestingly, for $\delta = 7$, the geometric bound (8) is considerably tighter than the Mohar bound (6), but is weaker than our bound (7). However, as $\delta \rightarrow \infty$, the geometric bound implies only the trivial inequality $h(\delta) \geq 0$.

We also establish the following upper bound:

$$h(\delta) \leq \sqrt{\frac{\delta - 6}{\delta - 2}} \quad (\delta \geq 7). \quad (9)$$

In fact, this bound is approached above by a lens sequence convergent to a δ -regular plane triangulation. Bound (7) is not tight; we conjecture the following equality.

Conjecture. The upper bound (9) is the exact value for $h(\delta)$ ($\delta \geq 7$).

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