

On the Yoneda completion of a quasi-metric space

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Abstract

Several theories aimed at reconciling the partial order and the metric space approaches to Domain Theory have been presented in the literature (e.g. [FK97], [BvBR98], [Smy89] and [Wag94]).

We focus in this paper on two of these approaches: the Yoneda completion of generalized metric spaces of [BvBR98], which finds its roots in work by Lawvere ([Law73], cf. also [Wag94]) and which is related to early work by Stoltenberg (e.g. [Sto67], [Sto67a] and [FG84]), and the Smyth completion ([Smy89],[Smy91],[Smy94],[Sün93] and [Sün95]).

A net-version of the Yoneda completion, complementing the net-version of the Smyth completion ([Sün95]), is given and a comparison between the two types of completion is presented.

The following open question is raised in [BvBR98]: “An interesting question is to characterize the family of generalized metric spaces for which [the Yoneda] completion is idempotent (it contains at least all ordinary metric spaces).”

We show that the largest class of quasi-metric spaces idempotent under the Yoneda completion is precisely the class of Smyth-completable spaces. A similar result has been obtained independently by B. Flagg and P. Sünderhauf in [FS96]¹. We present an entirely new proof of the result via concrete standard techniques and compare this approach with the more abstract categorical machinery of [FS96]. Our proof is based on a new characterization of Smyth-completeness of quasi-metric spaces in terms of sequences, which considerably simplifies prior characterizations for quasi-uniform spaces (e.g. [Sün93] and [Sün95]).

We also show that the ideal completion, and hence the Yoneda completion and the Smyth completion, are not sequentially adequate in general.

The study of the properties of total boundedness, precompactness, hereditary precompactness and compactness is motivated and we analyze the preservation of these properties under the two kinds of completion in the possible absence of idempotency.

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1 Background

The following notation is used throughout: \mathcal{N} denotes the set of natural numbers, \mathcal{R} denotes the set of real numbers and $\mathcal{R}^+ = [0, \infty)$.

For standard topological notions such as a topology, the supremum topology $\bigvee_{i \in I} \mathcal{T}_i$ of a family of topologies $(\mathcal{T}_i)_{i \in I}$ etc., we refer the reader to e.g. [Dug66].

If $(X, \mathcal{T}_1, \mathcal{T}_2)$ is a triple consisting of a set X and two topologies \mathcal{T}_1 and \mathcal{T}_2 on X , then the notation $\text{int}_{\mathcal{T}_i} A$, where $A \subseteq X$ and $i \in \{1, 2\}$, indicates the interior of the set A with respect to the topology \mathcal{T}_i .

A topology \mathcal{T} on a set X is T_0 if $\forall x, y \in X. x \neq y \Rightarrow \exists O \in \mathcal{T}. x \in O$ and $y \notin O$ or $\exists O' \in \mathcal{T}. x \notin O'$ and $y \in O'$.

Given a sequence $(x_n)_n$ in \mathcal{R} , let A be its set of points of accumulation. Then, with respect to the usual order on the reals, $\text{limsup}_n x_n$ is the least upperbound of A , provided it exists and $+\infty$ otherwise, and $\text{liminf}_n x_n$ is the greatest lowerbound of A , provided it exists and $-\infty$ otherwise. We use the notation " $\overline{\text{lim}}$ " and " $\underline{\text{lim}}$ " for limsup and liminf .

A *filter* \mathcal{F} on a set X is a non-empty subset of $\mathcal{P}(X)$ such that

- 1) $\forall F, G \in \mathcal{F}. F \cap G \in \mathcal{F}$
- 2) $\forall G \subseteq X. (\exists F \in \mathcal{F}. F \subseteq G) \Rightarrow G \in \mathcal{F}$
- 3) $\emptyset \notin \mathcal{F}$.

A *base* \mathcal{B} for a filter \mathcal{F} of $\mathcal{P}(X)$, is a subset of \mathcal{F} such that $\mathcal{F} = \{F \mid \exists B \in \mathcal{B}. B \subseteq F\}$.

For any set X , let $\Delta = \{(x, x) \mid x \in X\}$. For any two binary relations R and S , let $R \circ S$ denote their composition and let R^{-1} denote the inverse relation obtained from R .

A *quasi-uniformity* on a set X is a filter \mathcal{U} of $\mathcal{P}(X \times X)$ such that:

- 1) $\forall U \in \mathcal{U}. \Delta \subseteq U$
- 2) $\forall U \in \mathcal{U} \exists V \in \mathcal{U}. V \circ V \subseteq U$.

A *uniformity* on a set X is a quasi-uniformity \mathcal{U} on X which also satisfies:

- 1) $\forall U \in \mathcal{U}. U^{-1} \in \mathcal{U}$

The elements of a (quasi-)uniformity are referred to as *entourages*. A *(quasi-)uniform space* is a pair (X, \mathcal{U}) consisting of a set X and a (quasi-)uniformity \mathcal{U} on X . If (X, \mathcal{U}) is a quasi-uniform space, A a subset of X and $U \in \mathcal{U}$, then $U(A) = \bigcup \{U(x) \mid x \in A\}$, where $\forall x \in X. U(x) = \{y \mid (x, y) \in U\}$.

If \mathcal{U} is a quasi-uniformity on a set X then the *trace* quasi-uniformity $\mathcal{U}|A$ of \mathcal{U} on a subset A of X is defined by: $\mathcal{U}|A = \{U \cap (A \times A) \mid U \in \mathcal{U}\}$.

A function $f: (X, \mathcal{U}) \rightarrow (Y, \mathcal{V})$ between quasi-uniform spaces is *quasi-uniformly continuous* if $\forall V \in \mathcal{V} \exists U \in \mathcal{U}. f^2(U) \subseteq V$, where $f^2(U) = \{(f(x), f(y)) \mid xUy\}$. A

function $f: (X, \mathcal{U}) \rightarrow (Y, \mathcal{V})$ between quasi-uniform spaces is a *quasi-unimorphism* iff f is a bijection and f and f^{-1} are quasi-uniformly continuous.

The conjugate quasi-uniformity \mathcal{U}^{-1} of a quasi-uniformity \mathcal{U} is defined by: $\mathcal{U}^{-1} = \{U^{-1} \mid U \in \mathcal{U}\}$. Moreover \mathcal{U}^* is the coarsest uniformity on X finer than \mathcal{U} and \mathcal{U}^{-1} .

The topology $\mathcal{T}(\mathcal{U})$ induced by a quasi-uniformity \mathcal{U} on X is $\{O \subseteq X \mid \forall x \in X \exists U \in \mathcal{U}. U(x) \subseteq O\}$. A quasi-uniform space (X, \mathcal{U}) is T_0 if the topology $\mathcal{T}(\mathcal{U})$ is T_0 . The quasi-uniform space *generated by a partial order* (X, \leq) is the space (X, \mathcal{U}_{\leq}) where \mathcal{U}_{\leq} is the filter generated by the singleton $\{\leq\}$. We denote this by $\mathcal{U} = \{\leq\} \uparrow$. We remark that \mathcal{U}_{\leq} corresponds to the quasi-uniformity generated by the base $\mathcal{B} = \{\leq\}$ (e.g. [FL82]). We recall that the associated partial order of a T_0 quasi-uniform space (X, \mathcal{U}) is the relation $\leq_{\mathcal{U}} = \cap \mathcal{U}$. In case $\mathcal{U} = \{\leq_{\mathcal{U}}\} \uparrow$, we say that the quasi-uniform space *encodes* the partial order $\leq_{\mathcal{U}}$.

A quasi-uniform space (X, \mathcal{U}) is *precompact* if for each $U \in \mathcal{U}$ there is a finite subset F of X such that $X = U(F)$. A binary relation U on a set X is called *hereditarily precompact* provided that for any $A \subseteq X$ there is a finite set F such that $A \subseteq U(F)$. A quasi-uniform space is hereditarily precompact if each of its entourages is hereditarily precompact or, equivalently, each of its subspaces is precompact.

A *quasi-order* is a reflexive transitive binary relation. A *well-quasi-order* is a quasi-order in which every strictly decreasing sequence is finite and for which every set of pairwise incomparable elements is finite. One of the (many) equivalent definitions of the notion of a well-quasi-order is a quasi-order in which each non-empty set has at least one but not more than finitely many (non-equivalent) minimal elements.

A function $d: X \times X \rightarrow \mathcal{R}^+ \cup \{+\infty\}$ is a *quasi-pseudo-metric* if

- 1) $\forall x. d(x, x) = 0$
- 2) $\forall x, y, z. d(x, y) + d(y, z) \geq d(x, z)$.

We remark that traditionally a quasi-pseudo-metric is required to take finite values (e.g. [FL82]). However as the Yoneda completion of [BvBR98] involves generalized metrics which allow for infinite distances, we have adapted the definition of a quasi-pseudo-metric accordingly. It is well known that this difference does not affect classical topological results concerning the spaces in general, and this is in particular the case for the results referred to in the present paper.

A *quasi-pseudo-metric space* (X, d) is a pair consisting of a set X and a quasi-pseudo-metric d on X .

The *topology* \mathcal{T}_d *associated with* a quasi-pseudo-metric space (X, d) is the topology generated by the base consisting of the sets $B_{\epsilon}[x] = \{y \mid d(x, y) < \epsilon\}$, where $\epsilon > 0$ and $x \in X$.

The *quasi-uniformity* \mathcal{U}_d *generated by a quasi-pseudo-metric* d on a set X is the filter generated on $X \times X$ by the set of relations $(B_{\epsilon > 0})_{\epsilon}$, where $\forall \epsilon > 0. B_{\epsilon} = \{(x, y) \mid d(x, y) < \epsilon\}$.

A quasi-pseudo-metric space is T_0 if its associated topology is T_0 .

If the quasi-pseudo-metric space (X, d) is T_0 then we refer to the space as a *quasi-metric* space. In that case axiom 1) and the T_0 -condition can simply be replaced by:

$$1') \forall x, y. d(x, y) = d(y, x) = 0 \Leftrightarrow x = y.$$

The *conjugate* d^{-1} of a quasi-pseudo-metric d is defined to be the function $d^{-1}(x, y) = d(y, x)$, which is again a quasi-pseudo-metric (e.g. [FL82]). The conjugate of a quasi-pseudo-metric space (X, d) is the quasi-pseudo-metric space (X, d^{-1}) . The *pseudo-metric* d^* induced by a quasi-pseudo-metric d is defined by $d^*(x, y) = \max\{d(x, y), d(y, x)\}$. The topology induced by the pseudo-metric d^* is referred to as the *associated symmetric topology*. The *associated preorder* \leq_d of a quasi-pseudo-metric d is defined by $x \leq_d y \Leftrightarrow d(x, y) = 0$.

If (X, d) is a quasi-metric space then we define the equivalence relation \approx_d by: $\forall x, y \in X. x \approx_d y \Leftrightarrow x \leq_d y$ and $y \leq_d x$.

We write that a quasi-metric space (X, d) *encodes* a partial order when $\forall x, y \in X. d(x, y) \in \{0, +\infty\}$. In that case we also write that the *encoded order* is the order (X, \leq_d) . Conversely, for a given partial order (X, \leq) , one can define a quasi-metric space (X, d_{\leq}) which encodes the order, in the obvious way.

The function $d_1: \mathcal{R}^2 \rightarrow \mathcal{R}^+$, defined by $d_1(x, y) = y - x$ when $x < y$ and $d_1(x, y) = 0$ otherwise, and its conjugate are quasi-pseudo-metrics. We refer to d_1 as the “left distance” and to its conjugate as the “right distance”. These quasi-pseudo-metrics correspond to the nonsymmetric versions of the standard metric m on the reals, where $\forall x, y \in \mathcal{R}. m(x, y) = |x - y|$.

Note that the right distance has the usual order on the reals as associated order, that is $\forall x, y \in \mathcal{R}. x \leq_{d_1^{-1}} y \Leftrightarrow x \leq y$, while for the left distance we have $\forall x, y \in \mathcal{R}. x \leq_{d_1} y \Leftrightarrow x \geq y$.

For the definition of a net, we refer the reader for instance to [Gie80]. A *Cauchy net* on a quasi-pseudo-metric space (X, d) is a net $(x_\lambda)_{\lambda \in \Lambda}$ such that $\forall \epsilon > 0 \exists \lambda_0 \forall \nu \geq \lambda_0. d(x_\mu, x_\nu) < \epsilon$. The terminology “forward (or left) Cauchy nets” is sometimes used to indicate Cauchy nets, as opposed to “backward (or right) Cauchy nets”; that is nets which are Cauchy with respect to d^{-1} . A *biCauchy net* on a quasi-pseudo-metric space (X, d) is a net $(x_\lambda)_{\lambda \in \Lambda}$ such that $\forall \epsilon > 0 \exists \lambda_0 \forall \mu, \nu \geq \lambda_0. d(x_\mu, x_\nu) < \epsilon$. A net $(x_\lambda)_{\lambda \in \Lambda}$ on (X, d) is biCauchy if the net $(x_\lambda)_{\lambda \in \Lambda}$ is a Cauchy net on the metric space (X, d^*) .

A function $f: (X, d) \rightarrow (X', d')$ is an *isometry* if f is a bijection and $\forall x, y \in X. d'(f(x), f(y)) = d(x, y)$. If there exists an isometry between the quasi-metric spaces (X, d) and (X', d') then we say that these spaces are *isometric* and we denote this by: $(X, d) \cong (X', d')$.

A quasi-pseudo-metric space (X, d) is *totally bounded* if $\forall \epsilon > 0 \exists x_1, \dots, x_n \in X \forall x \in X \exists i \in \{1, \dots, n\}. d^*(x, x_i) < \epsilon$.

A quasi-pseudo-metric space (X, d) is *precompact* if $\forall \epsilon > 0 \exists x_1, \dots, x_n \in X \forall x \in X \exists i \in \{1, \dots, n\}. d(x_i, x) < \epsilon$.

For metric spaces, the notion of precompactness and total boundedness coincide.

In general, one only has that total boundedness implies precompactness.

A quasi-pseudo-metric space (X, d) is *hereditarily precompact* when every subspace of the space (X, d) is precompact.

A quasi-uniform space (X, \mathcal{U}) is *bicomplete* if the uniform space (X, \mathcal{U}^*) is complete.

A *bicompletion* of a quasi-uniform space (X, \mathcal{U}) is a bicomplete quasi-uniform space (Y, \mathcal{V}) which has a $\mathcal{T}(\mathcal{U}^*)$ -dense subspace quasi-unimorphic to (X, \mathcal{U}) . T_0 quasi-uniform spaces have a unique (up to quasi-unimorphism) T_0 bicompletion ([FL82]), indicated by “the bicompletion”.

A quasi-metric space (X, d) is *bicomplete* if every biCauchy sequence converges with respect to \mathcal{T}_{d^*} . The (sequential) bicompletion $(\overline{X}^B, \overline{d}^B)$ of a quasi-metric space (X, d) is defined as follows (e.g. [DiC71]):

$$\begin{aligned}\overline{X}^b &= \{(x_n)_n \mid (x_n)_n \text{ is bi-Cauchy}\} \\ \overline{d}^b((x_n)_n, (y_n)_n) &= \lim_{n \rightarrow \infty} d(x_n, y_n). \\ \overline{X}^B &= \overline{X}^b / \approx_{\overline{d}^b} \\ \overline{d}^B([(x_n)_n], [(y_n)_n]) &= \overline{d}^b((x_n)_n, (y_n)_n).\end{aligned}$$

We remark that $\lim_{n \rightarrow \infty} d(x_n, y_n)$ is well defined in this context, since $(d(x_n, y_n))_n$ is a Cauchy sequence in \mathcal{R} with respect to the ordinary metric.

Finally, we recall the definition of the ideal completion (e.g. [DP90]) and of the chain completion.

If (P, \sqsubseteq) is a partial order and A is a nonempty subset of P , then A is an *ideal* if $\forall y \in A. x \sqsubseteq y \Rightarrow x \in A$ and A is directed; that is $\forall x, y \in A \exists z \in A. x \sqsubseteq z$ and $y \sqsubseteq z$.

A *cofinal subset* B of a directed partial order (A, \sqsubseteq) is a subset B of A which satisfies the following: $\forall x \in A \exists y \in B. x \sqsubseteq y$.

The *ideal completion* of a partial order (P, \sqsubseteq, \perp) with a least element \perp , is the partial order $(Q, \subseteq, \{\perp\})$ where Q is the set of all ideals.

We remark that in the following we will use the standard terminology “chain completion” as used in theoretical computer science (e.g. [BvBR98]), where the notion of a chain refers to a countable linear order. This replaces the standard mathematical definition of a chain as a linear order.

Let (P, \leq) be a partial order. A sequence $(x_n)_n$ in P is *eventually increasing* if $\exists n_0 \forall m, n \geq n_0. m \leq n \Rightarrow x_m \leq x_n$. We let $S(P)$ denote the set of eventually increasing sequences for this partial order.

The *chain completion* of a partial order (P, \leq) is defined to be the partial order $(\overline{P}^i, \sqsubseteq^i)$, where $\overline{P}^i = S(P)_{/\approx^i}$ and where:

$$\forall (x_n)_n \in S(P) \forall (y_m)_m \in S(P).$$

$$(x_n)_n \sqsubseteq^i (y_m)_m \Leftrightarrow \exists n \forall k \geq n \forall m \exists l \geq m. x_k \leq y_l$$

$$(x_n)_n \approx^i (y_m)_m \Leftrightarrow (x_n)_n \sqsubseteq^i (y_m)_m \text{ and } (y_m)_m \sqsubseteq^i (x_n)_n$$

$$\forall [(x_n)_n] \in \overline{P}^i \forall [(y_m)_m] \in \overline{P}^i.$$

$$[(x_n)_n] \sqsubseteq^i [(y_m)_m] \Leftrightarrow (x_n)_n \sqsubseteq^i (y_m)_m.$$

Remarks:

- 1) It is easy to verify that the relation \approx is an equivalence relation.
- 2) It is well known (e.g. [BvBR98]) that for countable partial orders, the ideal completion and the chain completion coincide.
- 3) An equivalent version of the chain completion encountered in the literature (e.g. [CV74]) is the following:

The chain completion of a partial order (P, \leq) is defined to be the pair $(\overline{P}, \sqsubseteq)$, where $\overline{P} = S(P)_{/\approx}$ and where:

$$\forall (x_n)_n \in S(P) \forall (y_m)_m \in S(P).$$

$$(x_n)_n \sqsubseteq (y_m)_m \Leftrightarrow \exists k' \forall k \geq k' \exists l' \forall l \geq l'. x_k \leq y_l$$

$$(x_n)_n \approx (y_m)_m \Leftrightarrow (x_n)_n \sqsubseteq (y_m)_m \text{ and } (y_m)_m \sqsubseteq (x_n)_n$$

$$\forall [(x_n)_n] \in \overline{P} \forall [(y_m)_m] \in \overline{P}.$$

$$[(x_n)_n] \sqsubseteq [(y_m)_m] \Leftrightarrow (x_n)_n \sqsubseteq (y_m)_m.$$

It is this last version of the chain completion which has led to the Smyth completion, as apparent from the sequential version of the Smyth completion presented in [Smy89].

2 Introduction

Two main types of domain theoretic completions are known: the Smyth completion ([Smy89],[Smy91],[Smy94],[Sün93] and [Sün95]) and the Yoneda completion [BvBR98].

One can verify that the Smyth completion is an idempotent operation, while the Yoneda completion is not. We recall that the idempotency of the Smyth completion follows from the universal property of the completion, as remarked in [Sün93] (Section 3.3).

Historically, a sequential completion of quasi-uniform spaces, which has inspired the development of the Smyth completion, has first been introduced in [Smy89]. A different type of sequential completion considered in [Smy91] has led to the study of the Yoneda completion in [BvBR98]. Since this last completion has been formulated in terms of sequences, we will refer to it in the following as the “*sequential Yoneda completion*”.

We introduce a general net-version for the Yoneda completion, simply referred to as the “Yoneda completion”, complementing the net-version of the Smyth completion provided in [Sün95]. The two types of completion are compared and are shown to yield identical quasi-uniform spaces on any quasi-metric space.

In the remainder of the paper we focus on idempotency, sequential adequacy and on the preservation of topological properties by the completions.

Both the Yoneda completion and the Smyth completion have a definite appeal from a domain theoretic perspective, in view of the types of completion they embody. On the other hand each approach has characteristic problems.

In order to simplify the theory of the Smyth completion, Smyth has proposed in [Smy91] to focus on totally bounded spaces as domains of computation. As pointed out in [BvBR98], the price to be paid for the resulting simplification is that one has to work with a restricted class of spaces: the spectral spaces (e.g. [Smy91] and [Vic89]). Hence a full reconciliation between metric spaces and partial orders is not possible as only algebraic cpo’s which are so-called 2/3 SFP are spectral when taken with the Scott topology ([Plo83], [Smy91], [Smy92] and [Vic89]).

On the other hand, non-idempotent completions have not extensively been studied in the literature and hence their properties are not yet well known. This of course does not imply any negative properties of such a completion.

The non-idempotency of the Yoneda completion leads to the question, raised in [BvBR98], to characterize the class of spaces for which the Yoneda completion is idempotent. From the point of view of competitiveness with the approach presented in [Smy91], it would be desirable that this class is sufficiently large to at least include the totally bounded spaces.

We show that the largest class idempotent under the Yoneda completion consists of the Smyth-completable spaces, where the Yoneda completion of a Smyth-completable quasi-metric space coincides with its bicompletion. In particular we obtain that the Yoneda completion is idempotent on the class of totally bounded spaces.

A similar result has been obtained earlier and independently by Flagg and Sünderhauf ([FS96]). The proof methods involved are however entirely different.

The approach taken in [FS96] relies on the theory of continuity spaces and hence involves the machinery of categories enriched in a value quantale ([Fla96]).

The authors essentially transpose results of [Sün93] on topological quasi-uniform spaces to the context of continuity spaces, by introducing the notion of a topological continuity space. This is a continuity space equipped with an extra topology which is not necessarily the topology associated with the distance function of the continuity space. This allows one to define a notion of Smyth completion in this context, similar to the Smyth completion of a topological quasi-uniform space.

Again similar to the case of topological quasi-uniform spaces, where each quasi-uniform space equipped with its associated topology as extra topology can be shown to be a topological quasi-uniform space, the authors show that any continuity space equipped with its associated topology as extra topology, is a topological continuity space (Proposition 9.1 of [FS96]).

Moreover, a notion of ideal completion is introduced, by transposing ideas on the Yoneda completion of [BvBR98] from the context of generalized metric spaces to the context of value quantales.

The authors then show (cf. Theorem 29 of [FS96]) that for the case of a *continuity space*, viewed as a topological continuity space, the ideal completion and the Smyth completion have underlying continuity spaces which are isomorphic.

Then they proceed to show that the double iteration of this ideal completion on a continuity space is idempotent if and only if the Scott- and Alexandroff topologies on the original continuity space coincide (Corollary 32 of [FS96]).

Finally, they note that the double iteration of the ideal completion on a continuity space is idempotent if and only if the Scott and Alexandroff topologies on the ideal completion coincide (Corollary 32 of [FS96]). This involves Theorem 29 of [FS96] as well as the idempotency of the Smyth completion shown in [Sün93]. Theorem 5 of [Sün95], which relates the coincidence of Scott and Alexandroff topologies for the Smyth completion with the Smyth-completability of the space, is then used to reach the conclusion that the Yoneda completion of a continuity space is idempotent if and only if the space is Smyth-completable (Corollary 33).

Our own approach, in comparison, is much more direct, as it does not rely on prior results regarding the Smyth completion² and does not involve the more abstract category theoretic machinery of [FS96].

We focus on the notion of Smyth-completability for quasi-metric spaces. We characterize the Smyth-completable quasi-metric spaces as the spaces for which every Cauchy *sequence* is biCauchy (Theorem 9). This result might indeed be expected in the light of [Sün93] where a similar characterization for quasi-uniform spaces is

²With the exception of one application of Proposition 19 in order to show that the Yoneda completion coincides with the bicompletion on the class of Smyth-completable spaces. Cf. the proof of Theorem 26. This is only a convenient short cut which can easily be eliminated via a direct proof if one so desires.

presented in terms of nets, if one uses an intuition based on symmetric topology. The proof however requires some subtlety since non-symmetry is involved and the resulting characterization is considerably simpler since sequences are much easier to handle than nets.

Then we show that a quasi-metric space is Smyth-completable if and only if its bicompletion is Smyth-completable (Proposition 10).

Finally, using this result, we proceed to show that the class of Smyth-completable space is the largest class of spaces on which the Yoneda completion is idempotent.

It is a natural question, from a topological point of view, to consider whether, under a suitable condition, a sequential completion is *adequate*; that is whether a general completion based on filters or nets is replaceable by such a sequential completion. If this is the case, we say that the general completion is sequentially adequate (under the given condition).

For instance, in [Dug66], it is shown that for a uniform space with a countable base, sequential completions are adequate. The question is also relevant from a computer science point of view, since the possibility to work with sequences rather than with nets or filters leads to a considerable simplification of the theory.

We show that in general the ideal completion is not sequentially adequate and some implications for the sequential Yoneda completion of [BvBR98] are discussed.

From the fact that for Smyth-completable spaces the Yoneda completion reduces to the bicompletion, we obtain that the Yoneda completion preserves total boundedness and compactness with respect to the associated symmetric topology. As an exploration of how well behaved, from a topological point of view, the completion is, we analyze the preservation of topological properties which do not necessarily imply the idempotency of the completion. In particular we show that precompactness and compactness are preserved, but that hereditary precompactness is not. We remark that the choice for the study of these properties has in part been motivated by their connections with theoretical computer science.

A motivation for the study of totally bounded spaces has been given in [Smy91]. We investigate the preservation of precompactness as a non-symmetric version of this notion, for which the idempotency of the Yoneda completion is not guaranteed.

It is easy to see that precompactness for the case of a quasi-metric space (X, d) which encodes a partial order, amounts to the following requirement: $\exists x_1, \dots, x_n \in X. X = x_1 \uparrow \cup \dots \cup x_n \uparrow$, where $\forall x \in X. x \uparrow = \{y \in X \mid y \geq_d x\}$.

Since partial orders which arise in denotational semantics typically have a minimum \perp , the corresponding quasi-metric spaces are precompact. We remark that for bounded-complete ω -algebraic cpo's (e.g. [Smy91]), the notion of precompactness and the possession of a minimum coincide since any bounded-complete algebraic cpo is a join semilattice, that is any two elements have a greatest lower bound (e.g. [Gun92]).

This fact has been used in [Smy92] to guarantee that the notion of a spectral "cpo" of [Smy92], which does not require the existence of a minimum, truly is a cpo. In other words, the precompact spectral cpo's of [Smy92] are cpo's in the classical sense.

We recall that the notion of a well-quasi-order has well-known applications in the theory of rewrite systems, in particular in the context of termination proofs (e.g. [DJ90]). In [KR], the (strong) relationship between hereditary precompactness and well-quasi-orders is studied. In particular, it is shown that a binary relation is hereditarily precompact if and only if it is a well-quasi-order. Hence every hereditarily precompact quasi-uniform space which encodes a partial order, encodes a well-quasi-order.

Connections involving equational theories, rewrite rules, recursion and partial order completions have been discussed in the literature (e.g. [CV74]), raising the question as to whether hereditary precompactness and in particular the notion of a well-quasi-ordering, might be preservable by such completions. We answer this question in the negative for the Yoneda completion as well as for the Smyth completion, but remark that the property is preserved under the bicompletion. Finally, some motivation is given for the negative results in the context of termination proofs for rewrite systems.

For the sake of completeness, we briefly recall in the following section the definition of a topological quasi-uniform space as well as the definition of the Smyth completion of such a space. We also provide a simplified characterization of the notion of Smyth-completability.

3 The Smyth completion

We recall the definition of a topological quasi-uniform space (e.g. [Sün93] or [Kün93]).

Definition 1 *A topological quasi-uniform space is a triple $(X, \mathcal{U}, \mathcal{T})$ where X is a set, \mathcal{U} is a quasi-uniformity and \mathcal{T} is a topology on X such that the following axioms are satisfied:*

- (A₁) $\forall O \in \mathcal{T} \forall x \in O \exists U \in \mathcal{U} \exists O' \in \mathcal{T}. x \in O' \text{ and } U(O') \subseteq O.$
- (A₂) $\forall U \in \mathcal{U} \exists V \in \mathcal{U}. V \subseteq U \text{ and } \forall x \in X. V^{-1}(x) \text{ is } \mathcal{T}\text{-closed.}$
- (A₃) $\forall U \in \mathcal{U} \exists V \in \mathcal{U} \forall O \in \mathcal{T}. V(O) \subseteq \text{int}_{\mathcal{T}} U(O).$

A topological quasi-metric space is a triple (X, d, \mathcal{T}) consisting of a quasi-metric space (X, d) and a topology \mathcal{T} such that $(X, \mathcal{U}_d, \mathcal{T})$ is a topological quasi-uniform space.

Definition 2 *A TQUS-morphism $f: (X, \mathcal{U}, \mathcal{T}) \rightarrow (X', \mathcal{U}', \mathcal{T}')$ between topological quasi-uniform spaces is a map $f: X \rightarrow X'$ satisfying:*

- (C₁) *f is \mathcal{T} - \mathcal{T}' continuous.*
- (C₂) *f is \mathcal{U} - \mathcal{U}' quasi-uniformly continuous.*

Definition 3 Let $(X, \mathcal{U}, \mathcal{T})$ be a topological quasi-uniform space and let $U \in \mathcal{U}$. The relation $<_U$ of U -strong containment between subsets A and B of X is defined as follows:

$$A <_U B \Leftrightarrow \exists O, O' \in \mathcal{T}. A \subseteq O, U(O) \subseteq O' \text{ and } O' \subseteq B.$$

A filter \mathcal{F} of $\mathcal{P}(X)$ is S -Cauchy $\Leftrightarrow \forall U \in \mathcal{U} \forall F \in \mathcal{F} \exists x \in F \forall O \in \mathcal{T}. \{x\} <_U O \Rightarrow O \in \mathcal{F}$.

$$\text{A filter } \mathcal{F} \text{ of } \mathcal{P}(X) \text{ is round } \Leftrightarrow \forall F \in \mathcal{F} \exists O \in \mathcal{F} \cap \mathcal{T} \exists U \in \mathcal{U}. O <_U F.$$

The topological quasi-uniform space $(X, \mathcal{U}, \mathcal{T})$ is Smyth complete \Leftrightarrow every round S -Cauchy filter of $\mathcal{P}(X)$ is the \mathcal{T} -neighborhood filter of a unique point.

The definition of a computational Cauchy net ([Sün97]) will be useful in the following.

Definition 4 A computational Cauchy net $(x_\delta)_{\delta \in \Delta}$ on a topological quasi-uniform space $(X, \mathcal{U}, \mathcal{T})$ is a net such that for each entourage U there is an index $\delta_U \in \Delta$ such that for any $\delta' \in \Delta$ with $\delta' \geq \delta_U$ and for any $O \in \mathcal{T}$ with $x_{\delta'} <_U O$ we have that $(x_\delta)_{\delta \in \Delta}$ is eventually in O ; that is $\exists \delta_0 \forall \delta \geq \delta_0. x_\delta \in O$.

Definition 5 Let $(X, \mathcal{U}, \mathcal{T})$ be a topological quasi-uniform space. Define for arbitrary $A \subseteq X$: $\tilde{A} = \{\mathcal{F} \mid \mathcal{F} \text{ is a round } S\text{-Cauchy filter of } \mathcal{P}(X) \text{ satisfying } A \in \mathcal{F}\}$. For $U \in \mathcal{U}$, define \tilde{U} by: $\forall \mathcal{F}, \mathcal{G} \in \tilde{X}. (\mathcal{F}, \mathcal{G}) \in \tilde{U} \Leftrightarrow \forall O, O' \in \mathcal{T} \cap \mathcal{F}. O <_U O' \Rightarrow O' \in \mathcal{G}$.

Theorem 6 [Sün93] The set $\tilde{\mathcal{U}} = \{V \in \mathcal{P}(\tilde{X} \times \tilde{X}) \mid \exists U \in \mathcal{U}. \tilde{U} \subseteq V\}$ is a quasi-uniformity on \tilde{X} and the set $\{\tilde{O} \mid O \in \mathcal{T}\}$ is a base for a T_0 topology $\tilde{\mathcal{T}}$ on \tilde{X} . The triple $(\tilde{X}, \tilde{\mathcal{U}}, \tilde{\mathcal{T}})$ is a Smyth complete topological quasi-uniform space.

If the topology \mathcal{T} is T_0 , the mapping $i: X \rightarrow \tilde{X}$ defined by $x \rightarrow \mathcal{N}(x)$, where $\mathcal{N}(x)$ is the \mathcal{T} -neighborhood filter at x , is an injective TQUS-morphism and X and $i(X)$ are isomorphic as topological quasi-uniform spaces. Moreover, $i(X)$ is $\tilde{\mathcal{T}}$ -dense in \tilde{X} .

Next we recall the universal property for the Smyth completion, which in particular implies the idempotency of this completion.

Proposition 7 [Sün93] If $f: (X, \mathcal{U}, \mathcal{T}) \rightarrow (X', \mathcal{U}', \mathcal{T}')$ is a TQUS-morphism and $(X', \mathcal{U}', \mathcal{T}')$ is Smyth complete, then there exists exactly one extension $\tilde{f}: (\tilde{X}, \tilde{\mathcal{U}}, \tilde{\mathcal{T}}) \rightarrow (X', \mathcal{U}', \mathcal{T}')$ of f , that is there is exactly one TQUS-morphism $\tilde{f}: \tilde{X} \rightarrow X'$ satisfying $f = \tilde{f} \circ i$.

3.1 Smyth-completability

The Smyth-completable (topological) quasi-uniform spaces have been defined in [Sün93] as the (topological) quasi-uniform spaces of which the Smyth completion is again a quasi-uniform space; a condition which in general is violated as indicated in [Sün93].

Apart from forming a class with nice closure properties with respect to the Smyth completion, the Smyth-completable spaces can be interpreted to form a class of non-symmetric spaces which still possess an ‘‘inherent symmetry’’; that is to form a class of ‘‘weakly symmetric’’ spaces.

This interpretation is based on a characterization of Smyth-completable (topological) quasi-uniform spaces in terms of Cauchy nets (Theorem 5 of [Sün95]), which we discuss below.

We adopt this characterization in what follows as an alternative definition of the Smyth-completable spaces, as this approach does not require any reference to the more abstract context of the theory of topological quasi-uniform spaces nor a reference to the Smyth completion.

The definition given below is based on an adaptation of this characterization to the specific case of the quasi-pseudo-metric spaces, which suffices for our purposes.

Definition 8 *A quasi-pseudo-metric space (X, d) is Smyth-completable if every Cauchy net on (X, d) is biCauchy.*

The Smyth-completable quasi-pseudo-metric spaces are weakly symmetric in the sense that any net which is a Cauchy net with respect to the quasi-pseudo-metric d is also a Cauchy net with respect to the metric d^* .

The weakly symmetric nature of the spaces is illustrated by the fact that some properties of metric spaces generalize, under suitable hypotheses, to the context of Smyth-completable spaces. For instance Proposition 12 of [Kün93]: “A hereditarily precompact Smyth-completable (topological) quasi-uniform space is totally bounded”, which can be viewed as a generalization of the symmetric case where precompactness implies total boundedness.

The weak symmetry also lies at the basis of the fact that for Smyth-completable spaces, the Smyth completion simplifies to the bicompletion ([Sün95]).

We show that the Smyth-completeness condition can be simplified for the case of quasi-pseudo-metric spaces to a requirement on sequences rather than on nets. In fact a characterization in terms of sequences corresponds precisely to a requirement given by Smyth. In [Smy91] (Section 8.3) it is remarked that (for quasi-metric spaces) “... as soon as the notions of Cauchy sequences (and filters), limits and completeness come into play, however, the situation becomes rather chaotic: many conflicting versions can be found in the mathematical literature.”

In order to remedy the situation, [Smy91] focuses on totally bounded spaces for which the three main notions (Cauchy, right Cauchy and biCauchy) coincide.

The reason for this is that for totally bounded quasi-pseudo-metric spaces every Cauchy sequence is biCauchy (cf. the proof of Theorem 10 of [Smy91]). We show that this condition for quasi-metric spaces corresponds to Smyth-completeness (cf. also [Sch96]).

Theorem 9 *A quasi-pseudo-metric space is Smyth-completable if and only if every Cauchy sequence on the space is biCauchy.*

Proof: Let (X, d) be a quasi-pseudo-metric space. It suffices to show that when every Cauchy sequence on (X, d) is biCauchy, the space (X, d) is Smyth-completable.

We assume by way of contradiction that every Cauchy sequence on the space is biCauchy, but that the space is not Smyth-completable. Then there exists a Cauchy net $(x_\lambda)_{\lambda \in \Lambda}$ which is not biCauchy, that is

- (1) $\forall \epsilon > 0 \exists \lambda_0 \forall \nu \geq \mu \geq \lambda_0. d(x_\mu, x_\nu) < \epsilon.$
(2) $\exists \epsilon_0 > 0 \forall \lambda \exists \mu, \nu \geq \lambda. d(x_\mu, x_\nu) \geq \epsilon_0.$

Consider $(\epsilon_i)_{i \geq 1}$, a strictly decreasing sequence such that $\epsilon_1 < \epsilon_0$ and with limit 0. We define a sequence $(x_{\mu_j}, x_{\nu_j})_{j \geq 1}$ by induction as follows:

- a) For ϵ_1 , obtain λ_1 via (1), and obtain x_{μ_1}, x_{ν_1} via (2) for λ_1 .
b) Assume that $(x_{\mu_j}, x_{\nu_j})_{1 \leq j \leq k}$ has been defined.

For ϵ_{k+1} , obtain λ'_{k+1} via (1). Consider an index λ_{k+1} such that $\lambda_{k+1} \geq \lambda'_{k+1}$, μ_k, ν_k and obtain $x_{\mu_{k+1}}$ and $x_{\nu_{k+1}}$ via (2) for λ_{k+1} .

The reader may find it helpful to refer to Figure 1 below.

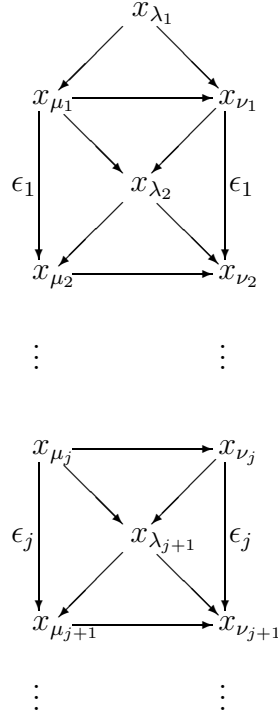


Figure 1

An arrow between points, say x_α and x_β , indicates that we measure the distance from x_α to x_β . Note that we only indicate arrows between points x_α and x_β , when $\alpha \leq \beta$. An index ϵ_i attached to an arrow indicates that the distance from x to y is less than ϵ_i .

The sequence $(x_{\mu_j}, x_{\nu_j})_{j \geq 1}$ satisfies the following property:

- (3) $\forall j \geq 1. d(x_{\mu_j}, x_{\nu_j}) \geq \epsilon_0.$

(Cf. the horizontal arrows of Figure 1.)

Note that in replacing λ'_{k+1} by a larger index λ_{k+1} , we have that (1) is still satisfied where λ_{k+1} dominates both μ_k and ν_k ; a property which is not necessarily guaranteed for λ'_{k+1} . This last fact together with the fact that $\forall j \geq 1. \mu_j, \nu_j \geq \lambda_j$ is represented by the diagonal arrows of Figure 1.

So we obtain the following inequalities: $\forall j \geq 1. \mu_{j+1}, \nu_{j+1} \geq \mu_j, \nu_j \geq \lambda_j$. Thus in particular, by (1), we have that the sequence $(x_{\mu_j}, x_{\nu_j})_{j \geq 1}$ satisfies the following properties:

- (4) $\forall j \geq 1. d(x_{\mu_j}, x_{\mu_{j+1}}) < \epsilon_j$ and $d(x_{\nu_j}, x_{\nu_{j+1}}) < \epsilon_j$
(5) $\forall j \geq 1. d(x_{\mu_j}, x_{\nu_{j+1}}) < \epsilon_j$ and $d(x_{\nu_j}, x_{\mu_{j+1}}) < \epsilon_j$.

(The vertical arrows of Figure 1 represent the inequalities displayed under (4).)

We remark that the sequence $(x_{\mu_j})_{j \in \mathcal{N}}$ is a Cauchy sequence. This follows from the choice of λ_{k+1} above, which is such that it dominates the index λ'_{k+1} obtained via (1) as well as the indices μ_k and ν_k .

We show that the sequence is not biCauchy. Note that $\forall j \geq 1. d(x_{\mu_{j+1}}, x_{\mu_j}) + d(x_{\mu_j}, x_{\nu_{j+1}}) \geq d(x_{\mu_{j+1}}, x_{\nu_{j+1}})$ and thus $d(x_{\mu_{j+1}}, x_{\mu_j}) \geq d(x_{\mu_{j+1}}, x_{\nu_{j+1}}) - d(x_{\mu_j}, x_{\nu_{j+1}})$.

By (3) and (5) we have that $d(x_{\mu_{j+1}}, x_{\mu_j}) \geq \epsilon_0 - \epsilon_j \geq \epsilon_0 - \epsilon_1$. So we obtain:

- (6) $\forall j \geq 1. d(x_{\mu_{j+1}}, x_{\mu_j}) \geq \epsilon$, where $\epsilon = \epsilon_0 - \epsilon_1$.

So the sequence $(x_{\mu_j})_{j \in \mathcal{N}}$ is not biCauchy. Thus we obtain a contradiction, which implies that the space (X, d) is Smyth-completable.

□

Examples of Smyth-completable quasi-metric spaces are the totally bounded quasi-metric spaces (e.g. [Sün95]) and the weightable quasi-metric spaces (e.g. [Kün93]). We remark that the fact that weightable quasi-metric spaces, or the equivalent partial metric spaces (e.g. [Mat94]) are Smyth-completable, can be obtained as a corollary of Theorem 2 (cf. [Sch97]).

Proposition 10 *If (X, d) is a quasi-pseudo-metric space then (X, d) is Smyth-completable if and only if its bicompletion $(\overline{X}^B, \overline{d}^B)$ is Smyth-completable.*

Proof: Suppose given a quasi-metric space (X, d) . If (X, d) is not Smyth-completable then there exists a Cauchy sequence $(x_n)_n$ in X which is not biCauchy. We recall (e.g. [DiC71]) that (X, d) is isometrically embedded in its bicompletion $(\overline{X}^B, \overline{d}^B)$, via the function $i: X \rightarrow \overline{X}^B$, where $\forall x \in X. i(x) = \overline{x}$ and where \overline{x} is the equivalence class in \overline{X}^B obtained by choosing as representative the sequence with constant value x . So we obtain that the sequence $(\overline{x}_n)_n$ is a Cauchy sequence of \overline{X}^B which is not biCauchy. Hence $(\overline{X}^B, \overline{d}^B)$ is not Smyth-completable.

To show the converse, by way of contradiction we assume that there is a Smyth-completable quasi-pseudo-metric space (X, d) for which the bicompletion $(\overline{X}^B, \overline{d}^B)$ is not Smyth-completable. So there exists a sequence $([(x_n^k)_n]_k)$ in \overline{X}^B which is Cauchy but not right Cauchy.

Hence we have:

$$(1) \forall \epsilon > 0 \exists n \forall l \geq k \geq n. \overline{d}^B([(x_n^k)_n], [(x_n^l)_n]) < \epsilon.$$

$$(2) \exists \epsilon_0 \forall n \exists l \geq k \geq n. \overline{d}^B([(x_n^l)_n], [(x_n^k)_n]) \geq \epsilon_0.$$

To obtain a contradiction, we construct a sequence $(y_n)_n$ in X which is Cauchy and not right Cauchy.

By (1), for each $K \geq 1$ we obtain an m_K such that:

$$(1') \forall l \geq k \geq m_K \exists N_1^K \forall n \geq N_1^K. d(x_n^k, x_n^l) < \frac{1}{3K}.$$

Then by (2) we obtain indices l_K and k_K for m_K such that $l_K \geq k_K \geq m_K$ and

$$(2') \exists N_2^K \forall n \geq N_2^K. d(x_n^{l_K}, x_n^{k_K}) \geq \frac{\epsilon_0}{2}.$$

We choose $M_K \geq \max\{N_1^K, N_2^K\}$, where N_1^K is obtained via (1') for l_K and k_K , and where M_K is large enough such that

$$(3') \forall n_1, n_2 \geq M_K. d(x_{n_1}^{l_K}, x_{n_2}^{l_K}) < \frac{1}{3K} \text{ and } d(x_{n_1}^{k_K}, x_{n_2}^{k_K}) < \frac{1}{3K}.$$

The last two inequalities can be obtained due to the fact that $(x_n^{l_K})_n$ and $(x_n^{k_K})_n$ are biCauchy sequences.

Without loss of generality we can assume that $\forall K \geq 1. M_K \leq M_{K+1}$. We can also assume that $\forall K \geq 1. l_K \leq k_{K+1}$ since we can choose m_{K+1} such that $m_{K+1} > l_K$.

We define the sequence $(y_n)_n$ as follows: $y_{2K-1} = x_{M_K}^{k_K}$ and $y_{2K} = x_{M_K}^{l_K}$.

By (2') it is clear that the sequence $(y_n)_n$ is not right Cauchy. We show that the sequence is Cauchy.

That is, we need to show that $\forall \epsilon > 0 \exists n_0 \forall j \geq i \geq n_0. d(y_i, y_j) < \epsilon$.

We use the following notation: let $y_i = x_{M_i}^{p_i}$ and $y_j = x_{M_j}^{q_j}$, where $p_i \in \{l_i, k_i\}$ and $q_j \in \{l_j, k_j\}$.

It suffices to remark that $\forall j \geq i \geq 1. d(y_i, y_j) = d(x_{M_i}^{p_i}, x_{M_j}^{q_j}) \leq d(x_{M_i}^{p_i}, x_{M_j}^{p_i}) + d(x_{M_j}^{p_i}, x_{M_j}^{q_j}) < \frac{1}{3i} + \frac{1}{3i}$, where the first term of the sum is obtained via (3') and the second term via (1').

□

Comment: We remark that the Smyth-completeness condition is necessary. An example of a bicomplete space which is not Smyth-completable is given in [Sün95].

4 The Yoneda completion

We present the straightforward definition of the Yoneda completion of a quasi-metric space without the category theoretic machinery of [BvBR98], as this suffices for our purposes.

Our terminology differs from the one used in [BvBR98] in that we refer to quasi-pseudo-metric spaces, while [BvBR98] refers to generalized metric spaces. We remark that these notions are equivalent, provided that infinite distances are allowed for the case of quasi-pseudo-metrics. Since there are currently alternative generalizations of metric spaces used in theoretical computer science, aside from quasi-pseudo-metric spaces (e.g. [Smy95]), we opt for the standard topological terminology of e.g. [FL82] where we allow for infinite distances (cf. Section 1).

We will state the definition of the Yoneda completion of a quasi-metric space; that is we require the T_0 -separation axiom to hold. T_0 -separation is of course a justifiable condition in Computer Science and thus a sound condition in the context of [BvBR98]. Also, the fact that the Yoneda completion of a non- T_0 space is T_0 , is another motivation to consider the class of quasi-metric spaces.

We remark that a central result of this section, Theorem 26, does not depend on this hypothesis; that is our characterization of the largest class on which the Yoneda completion is idempotent remains valid for general quasi-pseudo-metric spaces. The proof can be adapted by straightforward technical modifications.

Definition 11 *If (X, d) is a quasi-metric space, then an element x is the limit of the sequence $(x_n)_n \Leftrightarrow \forall y \in X. d(x, y) = \inf_n \sup_{k \geq n} d(x_k, y)$.*

The following definition of the Yoneda completion in a non-categorical form has been introduced first by M. Smyth in [Smy91]. Since this completion is stated in terms of sequences and since we will provide a more general version of the completion in terms of nets (cf. Subsection 4.1), we will refer to the completion defined below as the “*sequential Yoneda completion*”.

Definition 12 *A quasi-metric space is sequentially Yoneda complete if every Cauchy sequence has a limit.*

The sequential Yoneda completion of a quasi-metric space (X, d) is the pair $(\overline{X}^Y, \overline{d}^Y)$ obtained as follows:

$$\begin{aligned} \overline{X}^y &= \{(x_n)_n \mid (x_n)_n \text{ is Cauchy}\} \\ \overline{d}^y((x_n)_n, (y_n)_n) &= \inf_n \sup_{k \geq n} \sup_m \inf_{l \geq m} d(x_k, y_l) \\ \overline{X}^Y &= \overline{X}^y / \approx_{\overline{d}^y} \\ \overline{d}^Y([(x_n)_n], [(y_n)_n]) &= \overline{d}^y((x_n)_n, (y_n)_n). \end{aligned}$$

As remarked in [BvBR98], limits in the context of quasi-pseudo-metric spaces have distance 0 in general and thus are unique for quasi-metric spaces.

We remark that the sequential Yoneda completion of a quasi-metric space, as discussed in [BvBR98], is in fact a left version of a construction which already has been discussed by Stoltenberg in 1967 (e.g. [Sto67], [Sto67a] and [FG84]). Stoltenberg's completion uses a similar construction based on right Cauchy sequences and is one of the first non-symmetric completions discussed in the literature.

No direct comparison is however possible between the two types of completion, as Stoltenberg aimed to obtain an idempotent completion and the right Cauchy version of the sequential Yoneda completion is as such only part of the original construction for the Stoltenberg completion.

We illustrate below that the sequential Yoneda completion is not idempotent in general (as remarked in [BvBR98]). The reader familiar with the ideal completion may wish to omit this discussion.

We remark that it is easy to verify that when (X, d) encodes a partial order, this is also the case for $(\overline{X}^Y, \overline{d}^Y)$.

We define the sequential Yoneda completion of a partial order (X, \leq) to be the partial order $(\overline{X}^Y, \leq_{\overline{d}^Y})$, which we denote in what follows by: $(\overline{X}^Y, \overline{\leq}^Y)$.

In that case we have:

$$\begin{aligned} \overline{X}^Y &= \{(x_n)_n \mid \exists n_0 \forall m \geq n \geq n_0. x_n \leq x_m\} / \approx_{\overline{d}^Y} \\ (x_n)_n \overline{\leq}^Y (y_m)_m &\Leftrightarrow \exists n \forall k \geq n \forall m \exists l \geq m. x_k \leq y_l \end{aligned}$$

The last equivalence follows since: $(x_n)_n \overline{\leq}^Y (y_m)_m \Leftrightarrow \overline{d}_{\leq}^y((x_n)_n, (y_m)_m) = 0 \Leftrightarrow \overline{\lim}_n \underline{\lim}_m d_{\leq}(x_n, y_m) = 0 \Leftrightarrow \exists n \forall k \geq n \forall m \exists l \geq m. d_{\leq}(x_k, y_l) = 0$.

Given the partial order (\mathcal{N}, \leq) , then $\overline{\mathcal{N}}^y$ consists of the eventually increasing sequences of natural numbers. One can also easily verify that $(\overline{\mathcal{N}}^y, \overline{\leq}^Y) = \{\overline{0}, \dots, \overline{n}, \dots, \overline{\omega}\}$ and $(\overline{\overline{\mathcal{N}}^y}, \overline{\leq}^Y) = \{\overline{0}, \dots, \overline{\overline{n}}, \dots, \overline{\overline{\omega}}, \overline{\overline{\omega} + 1}\}$. Here, for every natural number n , \overline{n} is the equivalence class of all eventually increasing sequences which are eventually constant with value n and $\overline{\omega}$ is the class of all eventually increasing sequences which are not eventually constant. Similarly, for every natural number n , $\overline{\overline{n}}$ is the equivalence class of all eventually increasing sequences which are eventually constant with value \overline{n} , $\overline{\overline{\omega}}$ is the class of all eventually increasing sequences which are not eventually constant and $\overline{\overline{\omega} + 1}$ is the class of all eventually increasing sequences which are eventually constant with value $\overline{\omega}$.

So $(\overline{\overline{\mathcal{N}}^y}, \overline{\leq}^Y)$ is not order isomorphic to $(\overline{\mathcal{N}}^y, \overline{\leq}^Y)$ and thus the quasi-metric spaces which encode these partial orders are not isometric.

In the following subsection we introduce a net-version of the sequential Yoneda completion. This completion will simply be referred to as the “*Yoneda completion*”. We will show that the Yoneda completion of a quasi-metric space and the Smyth completion of a quasi-metric space, where the additional topology on the last completion

is omitted, coincide (cf. [FS96], Theorem 29, for a related result). The completeness of the Yoneda completion is derived from this equivalence result.

4.1 The Yoneda completion in terms of nets

In defining the Yoneda completion in terms of nets, we will frequently refer to the following collection of *all* Cauchy nets: $\overline{X}^y = \{(x_\gamma)_{\gamma \in \Gamma} \mid (x_\gamma)_{\gamma \in \Gamma} \text{ is a Cauchy net}\}$.

The reader familiar with set theory will note that this is a proper class, that is a class which is not a set. This is of course not a major problem, since “functions” still can be defined in this context. In particular the function considered in the lemma below.

Once we define the Yoneda completion (cf. Definition 15), we will only consider a quotient of this class, which will be a set as shown in a remark made at the end of this section.

In order to show that the usual quasi-metric on sequences can be extended to a quasi-metric on nets, we need the following lemma.

Lemma 13 *If (X, d) is a quasi-metric space and $\overline{X}^y = \{(x_\gamma)_{\gamma \in \Gamma} \mid (x_\gamma)_{\gamma \in \Gamma} \text{ is a Cauchy net}\}$ then $\overline{d}^y((x_\gamma)_{\gamma \in \Gamma}, (y_\delta)_{\delta \in \Delta}) = \inf_\gamma \sup_{\sigma \geq \gamma} \sup_\delta \inf_{\psi \geq \delta} d(x_\sigma, y_\psi)$ is a quasi-pseudo-metric on \overline{X}^y .*

Proof: We show that \overline{d}^y is a quasi-pseudo-metric. The idea is essentially due to Stoltenberg [Sto67]. We sketch the argument.

Let $(x_\gamma)_{\gamma \in \Gamma}$ be a Cauchy net of (X, d) and let $\epsilon > 0$. By the Cauchy property of $(x_\gamma)_\gamma$ there is $\gamma_0 \in \Gamma$ such that for all $\gamma_1, \gamma_2 \in \Gamma$ with $\gamma_0 \leq \gamma_1 \leq \gamma_2$ we have $d(x_{\gamma_1}, x_{\gamma_2}) < \epsilon$. Thus $\lim_\zeta \inf_{\gamma_2 \geq \zeta} d(x_{\gamma_1}, x_{\gamma_2}) \leq \epsilon$ whenever $\gamma_1 \in \Gamma$ and $\gamma_0 \leq \gamma_1$. Therefore $\overline{d}^y((x_\gamma)_\gamma, (x_\gamma)_\gamma) \leq \epsilon$ and thus $\overline{d}^y((x_\gamma)_\gamma, (x_\gamma)_\gamma) = 0$.

Suppose that $(x_\gamma)_{\gamma \in \Gamma}$, $(y_\delta)_{\delta \in \Delta}$ and $(z_\sigma)_{\sigma \in \Sigma}$ are three Cauchy nets in (X, d) and let $a = \overline{d}^y((x_\gamma)_\gamma, (y_\delta)_\delta)$ and $b = \overline{d}^y((y_\delta)_\delta, (z_\sigma)_\sigma)$. The triangle inequality for the situation under consideration holds if a or b are infinite. So we suppose that both are finite.

Let $\epsilon > 0$. Then $\overline{d}^y((x_\gamma)_\gamma, (y_\delta)_\delta) < a + \frac{\epsilon}{2}$ implies that there is a $\gamma_0 \in \Gamma$ such that $\lim_\zeta \inf_{\delta \geq \zeta} d(x_\gamma, y_\delta) < a + \frac{\epsilon}{2}$ whenever $\gamma \in \Gamma$ with $\gamma \geq \gamma_0$. Thus for each $\gamma \in \Gamma$ with $\gamma \geq \gamma_0$ there is a cofinal subset $\Delta(\gamma)$ of Δ such that $d(x_\gamma, y_\delta) < a + \frac{\epsilon}{2}$ whenever $\delta \in \Delta(\gamma)$.

Similarly, there is $\delta_0 \in \Delta$ such that for each $\delta \in \Delta$ satisfying $\delta \geq \delta_0$ there is a cofinal subset $\Sigma(\delta)$ of Σ with $d(y_\delta, z_\sigma) < b + \frac{\epsilon}{2}$ whenever $\sigma \in \Sigma(\delta)$. Thus for each $\gamma \in \Gamma$ such that $\gamma \geq \gamma_0$ there is $\delta_\gamma \in \Delta(\gamma)$ such that $\delta_\gamma \geq \delta_0$. Then $d(x_\gamma, y_{\delta_\gamma}) < a + \frac{\epsilon}{2}$ and $d(y_{\delta_\gamma}, z_\sigma) < b + \frac{\epsilon}{2}$ whenever $\sigma \in \Sigma(\delta_\gamma)$.

Consequently for each $\gamma \in \Gamma$ such that $\gamma \geq \gamma_0$ there is a cofinal subset $\Sigma(\delta_\gamma)$ of Σ , which we denote by $\Sigma(\gamma)$, such that $d(x_\gamma, z_\sigma) < a + b + \epsilon$ whenever $\sigma \in \Sigma(\gamma)$. We deduce that $\lim_\zeta \inf_{\sigma \geq \zeta} d(x_\gamma, z_\sigma) \leq a + b + \epsilon$ whenever $\gamma \in \Gamma$ such that $\gamma \geq \gamma_0$. Thus $\overline{d}^y((x_\gamma)_\gamma, (z_\sigma)_\sigma) \leq a + b + \epsilon$. We conclude that \overline{d}^y satisfies the triangle inequality.

□

Definition 14 *If (X, d) is a quasi-metric space, then an element x is the limit of the net $(x_\gamma)_{\gamma \in \Gamma} \Leftrightarrow \forall y \in X. d(x, y) = \inf_\gamma \sup_{\delta \geq \gamma} d(x_\delta, y)$.*

The following definition introduces the Yoneda completion of a quasi-metric space.

Definition 15 *A quasi-metric space is Yoneda complete if every Cauchy net has a limit.*

The Yoneda completion of a quasi-metric space (X, d) is the pair $(\overline{X}^Y, \overline{d}^Y)$ obtained as follows:

$$\begin{aligned} \overline{X}^y &= \{(x_\gamma)_{\gamma \in \Gamma} \mid (x_\gamma)_{\gamma \in \Gamma} \text{ is a Cauchy net}\} \\ \overline{d}^y((x_\gamma)_{\gamma \in \Gamma}, (y_\delta)_{\delta \in \Delta}) &= \inf_\gamma \sup_{\sigma \geq \gamma} \sup_\delta \inf_{\psi \geq \delta} d(x_\sigma, y_\psi) \\ \overline{X}^Y &= \overline{X}^y / \approx_{\overline{d}^y} \\ \overline{d}^Y([(x_\gamma)_{\gamma \in \Gamma}], [(y_\delta)_{\delta \in \Delta}]) &= \overline{d}^y((x_\gamma)_{\gamma \in \Gamma}, (y_\delta)_{\delta \in \Delta}). \end{aligned}$$

We leave it to the reader to verify that the above completion reduces to the ideal completion for the case of quasi-metric spaces which encode a partial order.

An indirect proof can also be given, using our Proposition 19 and the fact that the Smyth completion of a quasi-metric space which encodes a partial order reduces to the ideal completion ([FS96]).

Lemma 16 $\forall (x_\gamma)_{\gamma \in \Gamma}, (y_\delta)_{\delta \in \Delta} \in \overline{X}^y. \overline{d}^y((x_\gamma)_{\gamma \in \Gamma}, (y_\delta)_{\delta \in \Delta}) = \overline{\lim}_\gamma \underline{\lim}_\delta d(x_\gamma, y_\delta)$.

We omit the straightforward verification.

We recall some useful notions from [BvBR98]. We also introduce the notion of algebraicity familiar from e.g. [Smy91]. A non-equivalent version of algebraicity is presented in [BvBR98]. It is easy to verify that this last version, for the case of a quasi-metric encoding a partial order, reduces to our notion of sequential adequacy (cf. Definition 31 below). The fact that the two notions of algebraicity are not equivalent is illustrated by the comment preceding Lemma 33 below.

Definition 17 *An element e of a quasi-metric space (X, d) is finite if for each Cauchy net $(x_\gamma)_\gamma$ in X such that $(x_\gamma)_\gamma$ has a limit, we have:*

$$d(e, \lim_\gamma x_\gamma) = \sup_\gamma \inf_{\delta \geq \gamma} d(e, x_\delta).$$

A basis for a quasi-metric space (X, d) is a set B of finite elements, such that each element of X is the limit, with respect to the Yoneda completion, of a sequence of elements from B .

A quasi-metric space (X, d) is algebraic if each element from the space is a limit of a Cauchy net of finite elements.

A quasi-metric space (X, d) is ω -algebraic if the space is algebraic and has (at most) countably many finite elements.

The following lemma allows one to replace the equality in the definition of a finite element by an inequality and generalizes Proposition 3.4 of [BvBR98].

Lemma 18 *If $(x_\gamma)_\gamma$ is a Cauchy net in a quasi-metric space (X, d) such that $(x_\gamma)_\gamma$ has a limit then:*

$$\forall y \in X. d(y, \lim_\gamma x_\gamma) \leq \sup_\gamma \inf_{\delta \geq \gamma} d(y, x_\delta).$$

Proof: The proof is a straightforward adaptation of the proof of Proposition 3.4 of [BvBR98].

We present a sketch. If $(x_\gamma)_\gamma$ is a Cauchy net in a quasi-metric space (X, d) then: $\forall y \in X. d(y, \lim_\gamma x_\gamma) - \sup_\gamma \inf_{\delta \geq \gamma} d(y, x_\delta) = \inf_\gamma \sup_{\delta \geq \gamma} (d(y, \lim_\gamma x_\gamma) - d(y, x_\delta)) \leq \inf_\gamma \sup_{\delta \geq \gamma} d(x_\delta, \lim_\gamma x_\gamma) = d(\lim_\gamma x_\gamma, \lim_\gamma x_\gamma)$, where the last equality follows from the definition of a limit. So we obtain that $\forall y \in X. d(y, \lim_\gamma x_\gamma) - \sup_\gamma \inf_{\delta \geq \gamma} d(y, x_\delta) \leq 0$.

□

We compare the Yoneda completion with the Smyth completion in the following.

We will show that the results of completing a quasi-metric space by either type of completion, considered as quasi-uniform spaces, coincide. The reader may wish to compare this with a related result obtained in [FS96] (Theorem 29 of [FS96]).

As in Section 3 (Definition 5), we let \widetilde{X} denote the set of round S-Cauchy filters of the topological quasi-uniform space $(X, \mathcal{U}_d, \mathcal{T}(d))$.

We also use the following notation: if $(x_\gamma)_{\gamma \in \Gamma}$ is a net then $\mathcal{F}(x_\gamma)$ denotes the filter generated by the base consisting of the collection of sets $\{U(x_\gamma : \gamma \geq \delta) \mid U \in \mathcal{U} \text{ and } \delta \in \Gamma\}$. In case $(x_\gamma)_{\gamma \in \Gamma}$ is a Cauchy net, the filter $\mathcal{F}(x_\gamma)$ is a round S-Cauchy filter (cf. [Sün95]).

We shall identify in Proposition 19 each Cauchy net with its associated round S-Cauchy filter. Theorem 3 of [Sün95] ensures that each round S-Cauchy filter is of the form $\mathcal{F}(x_\delta)$ for some Cauchy net $(x_\delta)_{\delta \in \Delta}$. Our proof of Proposition 19 shows, among other things, that two Cauchy nets belong to the same equivalence class if and only if they are associated with the same round S-Cauchy filter: $\overline{d}^y((x_\delta)_{\delta \in \Delta}, (y_\lambda)_{\lambda \in \Lambda}) = \overline{d}^y((y_\lambda)_{\lambda \in \Lambda}, (x_\delta)_{\delta \in \Delta}) = 0$ if and only if $(\mathcal{F}(x_\delta), \mathcal{F}(y_\lambda))$ belongs to each $\widehat{U}_\epsilon \cap (\widehat{U}_\epsilon)^{-1}$.

Proposition 19 *The quasi-uniformity induced by \overline{d}^Y and the quasi-uniformity of the Smyth completion are equal on the collection of all round S-Cauchy filters.*

Proof: Let $\epsilon > 0$ and let $(x_\gamma)_{\gamma \in \Gamma}$ and $(y_\delta)_{\delta \in \Delta}$ be two Cauchy nets in (X, d) . First we show that if $(\mathcal{F}(x_\gamma), \mathcal{F}(y_\delta)) \in \widetilde{U}_\epsilon$, then for any $\kappa > 0$, we have $\overline{d}^y((x_\gamma)_\gamma, (y_\delta)_\delta) < \epsilon + 3\kappa$. Indeed, there is $\gamma_0 \in \Gamma$ such that for all $\mu \in \Gamma$ with $\mu \geq \gamma_0$ we have $\{x_\gamma | \gamma \geq \mu\} \subseteq U_\kappa(x_\mu)$, because $(x_\gamma)_\gamma$ is Cauchy. Fix such a $\mu \in \Gamma$. Since $U_\kappa(\{x_\gamma | \gamma \geq \mu\}) \in \mathcal{F}(x_\gamma)$, by definition of \widetilde{U}_ϵ we have $U_{\epsilon+\kappa}(\{x_\gamma | \gamma \geq \mu\}) \in \mathcal{F}(y_\delta)$. Thus there is $\sigma \in \Delta$ such that $\{y_\gamma | \gamma \geq \sigma\} \subseteq U_{\epsilon+\kappa}(\{x_\gamma | \gamma \geq \mu\}) \subseteq U_{\epsilon+2\kappa}(x_\mu)$. Therefore $\sup_\sigma \inf_{\delta \geq \sigma} d(x_\mu, y_\delta) \leq \epsilon + 2\kappa$. It follows that $\overline{d}^y((x_\gamma)_\gamma, (y_\delta)_\delta) < \epsilon + 3\kappa$.

Suppose now that $\overline{d}^y((x_\gamma)_{\gamma \in \Gamma}, (y_\delta)_{\delta \in \Delta}) < \epsilon$. We shall show that for each $\kappa > 0$, $(\mathcal{F}(x_\gamma), \mathcal{F}(y_\delta)) \in \widetilde{U}_{\epsilon+2\kappa}$. Let $A \in \mathcal{F}(x_\gamma)$. Then there are $\epsilon' > 0$ and $\nu \in \Gamma$ such that $U_{\epsilon'}(\{x_\gamma | \gamma \geq \nu\}) \subseteq A$ by definition of $\mathcal{F}(x_\gamma)$. There is $\gamma_0 \in \Gamma$ such that $\gamma_0 \geq \nu$ and $U_\epsilon(x_{\gamma_0})$ contains y_δ where δ runs through a cofinal subset of Δ , since $\overline{d}^y((x_\gamma)_\gamma, (y_\delta)_\delta) < \epsilon$. Then there is $\delta_0 \in \Delta$ such that $y_{\delta_0} \in U_\epsilon(x_{\gamma_0})$ and $d(y_{\delta_0}, y_\sigma) < \kappa$ whenever $\sigma \in \Delta$ such that $\sigma \geq \delta_0$, because $(y_\delta)_\delta$ is Cauchy. Therefore $U_\kappa(\{y_\sigma | \sigma \geq \delta_0\}) \subseteq U_{\epsilon+2\kappa}(x_{\gamma_0}) \subseteq U_{\epsilon+2\kappa}(A)$ and thus $U_{\epsilon+2\kappa}(A) \in \mathcal{F}(y_\delta)$ by definition of $\mathcal{F}(y_\delta)$. Hence $(\mathcal{F}(x_\gamma), \mathcal{F}(y_\delta)) \in \widetilde{U}_{\epsilon+2\kappa}$. □

The result proved above allows us to interpret the Yoneda completion of a quasi-pseudo-metric space as its Smyth completion by equipping it with its appropriate topology τ having the base $\{\widehat{O} | O \in \mathcal{T}(d)\}$ where $\mathcal{F}(x_\delta) \in \widehat{O}$ if and only if $O \in \mathcal{F}(x_\delta)$ (see [Sün93]).

A word of caution is necessary in order to avoid a possible misinterpretation of the above result. Proposition 19 does *not* imply that the Smyth completion and the Yoneda completion are equivalent in general nor does it imply a conflict between the different behaviors of the completions regarding idempotency!

We remark that Proposition 19 is stated for quasi-metric spaces (X, d) . Hence, for the case of the Smyth completion, the quasi-uniform space $(X, \mathcal{U}_d, \mathcal{T}(\mathcal{U}_d))$ associated with (X, d) will be viewed as a topological quasi-uniform space.

For this type of spaces, the two completions indeed give rise to identical quasi-uniform spaces, where for the case of the Smyth completion we obtain the quasi-uniform space from the completion by removing its second topology.

However, at a second iteration of the completions, the resulting spaces will be different in general. This follows from the fact that after performing the Smyth completion once on a given topological space of the kind described above, the resulting space will in general be a topological quasi-uniform space which is *not* a quasi-uniform space since the two topologies may differ (cf. [Sün93]). Hence the above result does not apply when carrying out the second iteration of the completions!

Remark: As mentioned above, the quotient of the proper class \overline{X}^y results in the set \overline{X}^Y . We recall that Proposition 19 implies that the Yoneda completion and the Smyth

completion of a quasi-metric space coincide. Combined with the fact that there is a one to one correspondence between the equivalence classes of the net-version of the Smyth completion and the round S-Cauchy filters on the space (cf. the comment preceding Proposition 19), we obtain that \overline{X}^Y is a set. Of course, the fact that the round S-Cauchy filters on the space form a set is an immediate consequence of the fact that each such filter is a subset of the powerset of this space.

Proposition 20 *If (X, d) is a quasi-metric space and $(\overline{X}^Y, \overline{d}^Y)$ is its Yoneda completion, then for any Cauchy net $(x_\gamma)_{\gamma \in \Gamma}$ we have that the limit of this net on the Smyth completion $(\widetilde{X}, \widetilde{\mathcal{U}}_d, \widetilde{\mathcal{T}}(\widetilde{\mathcal{U}}_d))$ is also the limit of this net for the Yoneda completion.*

Proof: Let (X, d) be a quasi-metric space for which the Smyth completion of the topological quasi-uniform space $(X, \mathcal{U}_d, \mathcal{T}(\mathcal{U}_d))$ is the topological quasi-uniform space $(\widetilde{X}, \widetilde{\mathcal{U}}_d, \widetilde{\mathcal{T}})$.

By $\mathcal{M}(x)$ we shall denote the round S-Cauchy filter on (X, d) representing the point $x \in \overline{X}^Y$.

Let $(x_\delta)_{\delta \in \Delta}$ be a Cauchy net in $(\overline{X}^Y, \overline{d}^Y)$. Then $(x_\delta)_{\delta \in \Delta}$ is a computational Cauchy net on $(\overline{X}^Y, \overline{d}^Y, \widetilde{\mathcal{T}})$. Therefore the derived filter $\mathcal{F}(x_\delta) = \{A \subseteq \overline{X}^Y : \text{there are } U \in \mathcal{U}(\overline{d}^Y) \text{ and } \delta_0 \in \Delta \text{ such that } \{x_\delta : \delta \in \Delta \text{ and } \delta \geq \delta_0\} <_U A\}$ is a round S-Cauchy filter (Proposition 4, [Sün97]). Since $(\overline{X}^Y, \overline{d}^Y, \widetilde{\mathcal{T}})$ is Smyth complete, $\mathcal{F}(x_\delta)$ is a $\widetilde{\mathcal{T}}$ -neighborhood filter $\mathcal{N}(x)$ for some unique point $x \in \overline{X}^Y$ ([Sün95]).

We want to show that $\overline{d}^Y(x, y) = \inf_\gamma \sup_{\delta \geq \gamma} \overline{d}^Y(x_\delta, y)$ whenever $y \in \overline{X}^Y$ in order to see that $(\overline{X}^Y, \overline{d}^Y)$ is complete. Fix $y \in \overline{X}^Y$.

We shall give an indirect proof of the fact that $\inf_\gamma \sup_{\delta \geq \gamma} \overline{d}^Y(x_\delta, y) \geq \overline{d}^Y(x, y)$.

By way of contradiction, we assume that $\inf_\gamma \sup_{\delta \geq \gamma} \overline{d}^Y(x_\delta, y) < \overline{d}^Y(x, y)$. Then there are $\rho, \epsilon > 0$ such that $\inf_\gamma \sup_{\delta \geq \gamma} \overline{d}^Y(x_\delta, y) < \rho$ and $\rho + 3\epsilon \leq \overline{d}^Y(x, y)$.

Furthermore there is $\gamma \in \Delta$ such that $\overline{d}^Y(x_\delta, y) < \rho$ whenever $\delta \in \Delta$ and $\delta \geq \gamma$; and thus for each $\delta \in \Delta$ with $\delta \geq \gamma$, $(\mathcal{M}(x_\delta), \mathcal{M}(y)) \in \widetilde{U}_{\rho+\epsilon}$ by the second part of the proof of Proposition 19, where we choose $\kappa = \frac{\epsilon}{2}$. Suppose that $O, O' \in \mathcal{T}(d)$ and $O <_{U_{\rho+\epsilon}} O'$ and $O \in \mathcal{M}(x)$. Then $x = \mathcal{M}(x) \in \widetilde{O}$. Thus there is $\delta_0 \geq \gamma$ such that $x_{\delta_0} = \mathcal{M}(x_{\delta_0}) \in \widetilde{O}$ and therefore $O \in \mathcal{M}(x_{\delta_0})$, since $(x_\delta)_{\delta \in \Delta}$ converges to x with respect to the topology $\widetilde{\mathcal{T}}$ because of $\mathcal{N}(x) = \mathcal{F}(x_\delta)$. Then by definition of $\widetilde{U}_{\rho+\epsilon}$, we see that $O' \in \mathcal{M}(y)$. Therefore by definition of $\widetilde{U}_{\rho+\epsilon}$ we conclude that $(\mathcal{M}(x), \mathcal{M}(y)) \in \widetilde{U}_{\rho+\epsilon}$.

Consequently by the first part of the proof of Proposition 19, where we choose $\kappa = \frac{\epsilon}{3}$, $\overline{d}^Y(x, y) < \rho + 2\epsilon < \overline{d}^Y(x, y)$ —a contradiction.

We conclude that $\inf_\gamma \sup_{\delta \geq \gamma} \overline{d}^Y(x_\delta, y) \geq \overline{d}^Y(x, y)$.

We proceed to show that

$$\inf_\gamma \sup_{\delta \geq \gamma} \overline{d}^Y(x_\delta, y) = \overline{d}^Y(x, y).$$

By way of contradiction, we assume that $\inf_{\gamma} \sup_{\delta \geq \gamma} \bar{d}^Y(x_{\delta}, y) > \bar{d}^Y(x, y)$. Then there is $\epsilon > 0$ such that $\inf_{\gamma} \sup_{\delta \geq \gamma} \bar{d}^Y(x_{\delta}, y) \geq 3\epsilon + \bar{d}^Y(x, y)$. For each $\delta \in \Delta$ choose $\delta'(\delta) \in \Delta$ such that $\delta'(\delta) \geq \delta$ and $\bar{d}^Y(x_{\delta'(\delta)}, y) \geq 2\epsilon + \bar{d}^Y(x, y)$. Because $(x_{\delta})_{\delta \in \Delta}$ is a Cauchy net and $\mathcal{F}(x_{\delta}) = \mathcal{N}(x)$ we see by Lemma 5 ([Sün97]) that $(x_{\delta})_{\delta \in \Delta}$ converges to x with respect to the topology $\mathcal{T}((\bar{d}^Y)^{-1})$. In particular there is $\delta_0 \in \Delta$ such that $\bar{d}^Y(x_{\delta'(\delta_0)}, x) < \epsilon$. It follows that $\bar{d}^Y(x, y) + 2\epsilon \leq \bar{d}^Y(x_{\delta'(\delta_0)}, y) \leq \bar{d}^Y(x_{\delta'(\delta_0)}, x) + \bar{d}^Y(x, y) < \epsilon + \bar{d}^Y(x, y)$ —a contradiction. We conclude that $\bar{d}^Y(x, y) = \inf_{\gamma} \sup_{\delta \geq \gamma} \bar{d}^Y(x_{\delta}, y)$ whenever $y \in \bar{X}^Y$.

□

Corollary 21 *The Yoneda completion (\bar{X}^Y, \bar{d}^Y) of a quasi-metric space (X, d) is (Yoneda) complete.*

Proof: This follows from Propositions 19 and 20.

□

Proposition 22 *There is an isometric embedding $i: (X, d) \rightarrow (\bar{X}^Y, \bar{d}^Y)$ defined by: $\forall x \in X. i(x) = [(x)_{\gamma}]$, where $(x)_{\gamma}$ is a net with constant value x . The set $i(X)$ is dense in \bar{X}^Y and consists of finite elements.*

Proof: We recall that by Proposition 19, $(\bar{X}^Y, \mathcal{U}_{\bar{d}^Y})$ and $(\widetilde{X}, \widetilde{\mathcal{U}})$ coincide. So by Proposition 11 of [Sün93] and Theorem 3 of [Sün95], we obtain that $i: (X, \mathcal{U}_d) \rightarrow (\bar{X}^Y, \mathcal{U}_{\bar{d}^Y})$ is a quasi-uniform embedding. It is easy to verify that $\forall x, y \in X. \bar{d}^Y(i(x), i(y)) = d(x, y)$ and hence i is an isometric embedding.

Again by Proposition 11 of [Sün93], we obtain that $i(X)$ is dense in \widetilde{X} with respect to $\widetilde{\mathcal{T}}(\widetilde{\mathcal{U}})$ and thus by Proposition 20, $i(X)$ is dense in \bar{X}^Y .

We show that the embedding $i(X)$ of the original space in its Yoneda completion (\bar{X}^Y, \bar{d}^Y) , consists of finite elements of the completion. We will underline elements of \bar{X}^Y in order to distinguish them from elements of X .

For any $x \in X$ we need to verify that a net $[(x)_{\gamma}]$, with constant value x , is finite.

To verify that $[(x)_{\gamma}]$ is finite, we need to show that for each Cauchy net $(\bar{y}_{\mu})_{\mu}$ in \bar{X}^Y the following equality holds:

$$\bar{d}^Y([(x)_{\gamma}], \lim_{\mu} \bar{y}_{\mu}) = \sup_{\mu} \inf_{\nu \geq \mu} \bar{d}^Y([(x)_{\gamma}], \bar{y}_{\nu}).$$

By Lemma 18 we know that $\bar{d}^Y([(x)_\gamma], \lim_\mu \bar{y}_\mu) \leq \sup_\mu \inf_{\nu \geq \mu} \bar{d}^Y([(x)_\gamma], \bar{y}_\nu)$.
By way of contradiction we assume that

$$\bar{d}^Y([(x)_\gamma], \lim_\mu \bar{y}_\mu) < \sup_\mu \inf_{\nu \geq \mu} \bar{d}^Y([(x)_\gamma], \bar{y}_\nu).$$

Hence there exists $\epsilon_0 > 0$ such that $\bar{d}^Y([(x)_\gamma], \lim_\mu \bar{y}_\mu) + \epsilon_0 \leq \sup_\mu \inf_{\nu \geq \mu} \bar{d}^Y([(x)_\gamma], \bar{y}_\nu)$.

We use the following notation for the limit and for representatives of the equivalence classes under consideration: let $\lim_\mu \bar{y}_\mu = \bar{y}$, $\bar{y} = [(y_\alpha)_\alpha]$ and $\bar{y}_\mu = [(y_\sigma^\mu)_\sigma]$.

Then we have: $\bar{d}^Y([(x)_\gamma], \lim_\mu \bar{y}_\mu) = \bar{d}^Y([(x)_\gamma], \bar{y}) = \inf_\gamma \sup_{\delta \geq \gamma} \sup_\alpha \inf_{\beta \geq \alpha} d(x, y_\beta)$
 $= \sup_\alpha \inf_{\beta \geq \alpha} d(x, y_\beta)$.

And: $\sup_\mu \inf_{\nu \geq \mu} \bar{d}^Y([(x)_\gamma], \bar{y}_\nu) = \sup_\mu \inf_{\nu \geq \mu} \inf_\gamma \sup_{\delta \geq \gamma} \sup_\sigma \inf_{\tau \geq \sigma} d(x, y_\tau^\nu) =$
 $\sup_\mu \inf_{\nu \geq \mu} \sup_\sigma \inf_{\tau \geq \sigma} d(x, y_\tau^\nu)$.

So $\sup_\alpha \inf_{\beta \geq \alpha} d(x, y_\beta) + \epsilon_0 \leq \sup_\mu \inf_{\nu \geq \mu} \sup_\sigma \inf_{\tau \geq \sigma} d(x, y_\tau^\nu)$.

Thus $\exists \mu_0 \forall \nu \geq \mu_0. \epsilon_0 \leq \sup_\sigma \inf_{\tau \geq \sigma} d(x, y_\tau^\nu) - \sup_\alpha \inf_{\beta \geq \alpha} d(x, y_\beta) =$
 $\inf_\alpha \sup_{\beta \geq \alpha} \sup_\sigma \inf_{\tau \geq \sigma} (d(x, y_\tau^\nu) - d(x, y_\beta)) \leq \inf_\alpha \sup_{\beta \geq \alpha} \sup_\sigma \inf_{\tau \geq \sigma} d(y_\beta, y_\tau^\nu) = \bar{d}^Y(\bar{y}, \bar{y}_\nu)$.

So we conclude that:

$$(I) \exists \mu_0 \forall \nu \geq \mu_0. \epsilon_0 \leq \bar{d}^Y(\bar{y}, \bar{y}_\nu).$$

We reach a contradiction by showing that:

$$(II) \forall \epsilon > 0 \forall \nu \exists \nu_1 \geq \nu. \bar{d}^Y(\bar{y}, \bar{y}_{\nu_1}) < \epsilon.$$

Let $\epsilon > 0$ and an index ν be given. Then, since $(\bar{y}_\mu)_\mu$ is a Cauchy net, we obtain for $\epsilon' = \frac{\epsilon}{2}$ an index ν_2 such that $\forall \mu_2 \geq \mu_1 \geq \nu_2. \bar{d}^Y(\bar{y}_{\mu_1}, \bar{y}_{\mu_2}) < \epsilon'$. We choose ν_1 to be an index such that $\nu_1 \geq \nu, \nu_2$.

Since $\lim_\mu \bar{y}_\mu = \bar{y}$, we obtain in particular that $\bar{d}^Y(\bar{y}, \bar{y}_{\nu_1}) = \inf_\mu \sup_{\nu \geq \mu} \bar{d}^Y(\bar{y}_\nu, \bar{y}_{\nu_1}) \leq$
 $\inf_{\mu \geq \nu_1} \sup_{\nu \geq \mu} \bar{d}^Y(\bar{y}_\nu, \bar{y}_{\mu_1}) \leq \epsilon'$.

Hence we obtain indeed that $\forall \epsilon \forall \nu \exists \nu_1 \geq \nu. \bar{d}^Y(\bar{y}, \bar{y}_{\nu_1}) \leq \epsilon' < \epsilon$.

To obtain the contradiction, we consider μ_0 obtained via (I) and then obtain $\nu_1 \geq \mu_0$ for $\epsilon = \frac{\epsilon_0}{2}$ via (II). This yields the contradiction: $\epsilon_0 \leq \bar{d}^Y(\bar{y}, \bar{y}_{\nu_1}) < \frac{\epsilon_0}{2}$.

□

Corollary 23 *The Yoneda completion of a quasi-metric space is algebraic.*

Proof: Let (X, d) be a quasi-metric space. We remark that for each Cauchy net $(x_\delta)_{\delta \in \Delta}$ in X , the embedded net $(i(x_\delta))_{\delta \in \Delta}$ in the Smyth completion converges to “itself”, i.e. to the point of the Smyth-completion with as representative the net $(x_\delta)_{\delta \in \Delta}$ (cf. [Sün93]). Hence, by Propositions 19 and 20, we obtain that the net also converges to itself, with respect to the Yoneda-completion.

The desired result now follows from the second part of Proposition 22.

□

4.2 Largest idempotent class

We recall the classical definition of idempotency (adapted to the specific case of the Yoneda completion).

Definition 24 *The Yoneda completion of a quasi-metric space (X, d) is idempotent $\Leftrightarrow \exists$ an isometry $i: \overline{X}^Y \rightarrow \overline{\overline{X}^Y}^Y$ such that $\forall \bar{x} \in \overline{X}^Y . i(\bar{x}) = \overline{\bar{x}}$.*

Definition 25 *The Yoneda completion is idempotent on a class of quasi-metric spaces $\mathcal{S} \Leftrightarrow$ 1) \mathcal{S} is closed under the Yoneda completion.
2) Every space in \mathcal{S} has an idempotent Yoneda completion.*

We focus on the following problem:

“Characterize the largest class (if any) of quasi-metric spaces on which the Yoneda completion is idempotent.”

The following theorem provides the answer (cf. also [FS96] for a similar result).

Theorem 26 *The class of the Smyth-completable quasi-metric spaces is the largest class of spaces on which the Yoneda completion is idempotent. The Yoneda completion and the Smyth completion reduce on this class to the bicompletion.*

Proof: Let \mathcal{S} denote the class of all Smyth-completable quasi-metric spaces.

We first verify that the Yoneda completion coincides with the bicompletion on \mathcal{S} .

Let (X, d) be a Smyth-completable quasi-metric space and let (X, \mathcal{U}_d) be the Smyth-completable quasi-uniform space associated with this quasi-metric space.

This quasi-uniform space can be interpreted as a topological quasi-uniform space in the usual way ([Sün93]); that is as the space: $(X, \mathcal{U}_d, \mathcal{T}(\mathcal{U}_d))$.

By Proposition 19 and the fact that the Smyth completion of a Smyth-completable quasi-uniform space reduces to the bicompletion ([Sün93]), we obtain that the quasi-uniform space associated with the Yoneda completion of (X, d) reduces to the bicompletion of (X, \mathcal{U}_d) . Since the bicompletion is sequentially adequate ([DiC71]), we can conclude that the Yoneda completion of a Smyth-completable quasi-metric space reduces to the sequential bicompletion of this quasi-metric space.

Since the bicompletion of a Smyth-completable space is Smyth-completable (Proposition 10), we obtain that \mathcal{S} is closed under the Yoneda completion.

Moreover, since $(\overline{X}^Y, \overline{d}^Y) \in \mathcal{S}$, we have that $(\overline{\overline{X}^Y}^Y, \overline{\overline{d}^Y}^Y)$ is the bicompletion of $(\overline{X}^Y, \overline{d}^Y)$ and thus $(\overline{\overline{X}^Y}^Y, \overline{\overline{d}^Y}^Y) \cong (\overline{X}^Y, \overline{d}^Y)$. Hence the Yoneda completion is idempotent on the class \mathcal{S} .

It remains to be shown that \mathcal{S} is the largest class with this property.

Assume by way of contradiction that there exists a quasi-metric space (X, d) which is not Smyth-completable, such that $i: (\overline{X}^Y, \overline{d}^Y) \cong (\overline{\overline{X}^Y}^Y, \overline{\overline{d}^Y}^Y)$, where $\forall \bar{x} \in \overline{X}^Y . i(\bar{x}) = \overline{\bar{x}}$.

By Theorem 9 there exists a Cauchy sequence $(x_n)_n$ of (X, d) which is not bi-Cauchy:

- (1) $\forall \epsilon > 0 \exists n_0 \forall n \geq m \geq n_0. d(x_m, x_n) < \epsilon$
- (2) $\exists \epsilon_0 > 0 \forall n \exists n_1 \geq m_1 \geq n. d(x_{n_1}, x_{m_1}) \geq \epsilon_0.$

First we note that

$$\begin{aligned} \overline{d}^Y([\overline{(x_n)_n}], [(\overline{x_n})_n]) &= \overline{\lim}_n \underline{\lim}_m \overline{d}^Y([(x_n)_n], \overline{x}_m) \\ &= \underline{\lim}_m \overline{d}^Y([(x_n)_n], \overline{x}_m) \\ &= \underline{\lim}_m \overline{\lim}_{n'} \underline{\lim}_{m'} d(x_{n'}, x_m) \\ &= \underline{\lim}_m \overline{\lim}_{n'} d(x_{n'}, x_m) \geq \epsilon_0. \end{aligned}$$

The first equality uses Lemma 16 and the fact that $[\overline{(x_n)_n}]$ is the sequence with constant value $[(x_n)_n]$. The second equality again uses this last fact, while the third equality once more uses Lemma 16 and also the fact that \overline{x}_m is the sequence with constant value x_m . Finally, the fourth equality follows from (2) above.

Thus we obtain that $\overline{d}^Y([\overline{(x_n)_n}], [(\overline{x_n})_n]) \neq 0$.

We verify that $[(x_n)_n]$ is a limit of the sequence $(\overline{x_n})_n$ in \overline{X}^Y .

We need to verify that $\forall [(y_n)_n] \in \overline{X}^Y. \overline{d}^Y([(x_n)_n], [(y_n)_n]) = \overline{\lim}_n \overline{d}^Y(\overline{x}_n, [(y_n)_n])$.

We remark that $\overline{d}^Y([(x_n)_n], [(y_n)_n]) = \overline{\lim}_n \underline{\lim}_m d(x_n, y_m)$, while $\overline{\lim}_n \overline{d}^Y(\overline{x}_n, [(y_n)_n]) = \overline{\lim}_n \overline{\lim}_{n'} \underline{\lim}_{m'} d(x_n, y_{m'}) = \overline{\lim}_n \underline{\lim}_{m'} d(x_n, y_{m'})$ and thus $\overline{d}^Y([(x_n)_n], [(y_n)_n]) = \overline{\lim}_n \overline{d}^Y(\overline{x}_n, [(y_n)_n])$.

Since $i: (\overline{X}^Y, \overline{d}^Y) \cong (\overline{X}^Y, \overline{d}^Y)$, where $i(\overline{x}) = \overline{\overline{x}}$, we immediately obtain that $[\overline{(x_n)_n}]$ is a limit of the sequence $(\overline{\overline{x_n}})_n$.

A similar calculation as above shows that $[(\overline{x_n})_n]$ is also a limit of the sequence $(\overline{\overline{x_n}})_n$.

Since we have uniqueness of limits, we obtain that $[(\overline{x_n})_n] = [\overline{(x_n)_n}]$ which is a contradiction since $\overline{d}^Y([\overline{(x_n)_n}], [(\overline{x_n})_n]) \neq 0$.

□

Remark: The proof also works in the absence of T_0 -separation, that is for quasi-pseudo-metric spaces, since in this case limits still have distance 0, which suffices to obtain the contradiction.

5 Sequential inadequacy

We show that the ideal completion is not sequentially adequate and discuss some implications for the Yoneda completion.

We recall the following well-known result for which the verification is straightforward.

Lemma 27 *For a countable partial order, the chain completion and the ideal completion coincide.*

We will show in the following that the ideal completion of a partial order, in general is not replaceable by a sequential completion; in other words, sequences are not adequate for the ideal completion. Hence the ideal completion is in general not replaceable by the chain completion. We will return to this remark at the end of the section where we discuss implications for sequential completions presented in the literature ([Smy89] and [BvBR98]).

We recall the definition of an (ω -)algebraic partially ordered set (e.g. [Mis91] and [Smy91]).

Definition 28 *An element e of a partially ordered set (P, \sqsubseteq) is finite if for each directed subset D of P for which $\sqcup D$ exists, $e \leq \sqcup D$ implies that $e \leq d$ for some $d \in D$. The set of finite elements of P is denoted by $F(P)$ and for each $x \in P$, $F(x) = \{e \in F(P) \mid e \sqsubseteq x\}$.*

An algebraic partially ordered set is a partially ordered set (P, \sqsubseteq) satisfying the property that for each $x \in P$, $F(x)$ is directed and that $x = \sqcup F(x)$.

An ω -algebraic partially ordered set (P, \sqsubseteq) is an algebraic partially ordered set such that $F(P)$ is at most countable.

Comment: It is easy to verify that for the case of a quasi-metric space (X, d) encoding a partial order, the ideal completion $(\overline{X}^Y, \overline{d}^Y)$ is such that for any ideal $\overline{x} \in \overline{X}^Y$, $F(\overline{x})$ is always directed.

Hence in this context the above notion of algebraicity and the notion of algebraicity of a quasi-metric space (cf. Definition 17) coincide.

We recall some of the basic theory of ordinals (e.g. [HJ84]). A set A is *transitive* if $\forall x, y \in A. x \in y \Rightarrow x \in A$. A set A is *well ordered by the membership relation* if (A, \in) is a total order and every nonempty subset of A has a least element. A set α is an *ordinal* if α is transitive and well ordered by the membership relation. The *successor* of an ordinal α is the ordinal $\alpha + 1$ defined by $\alpha + 1 = \alpha \cup \{\alpha\}$. An ordinal α is called a *successor ordinal* if $\alpha = \beta + 1$ for some ordinal β . Otherwise α is called a *limit ordinal*. Limit ordinals can be characterized as the ordinals α such that $\cup \alpha = \alpha$. An ordinal α is a cardinal when there does not exist a bijection between α and one of its elements.

We denote the first uncountable ordinal by ω_1 . The empty set \emptyset is an ordinal denoted by 0.

We recall that the partial ordering on an ordinal α is the subset-order \subseteq and that each ordinal consists of the ordinals which strictly precede it in the subset-order, where we have that $\alpha \subset \beta \Leftrightarrow \alpha \in \beta$.

Since every ordinal α is a total order, it is clear that every subset A of α is directed. Also, every subset A of α possesses a supremum which is $\cup A$.

Lemma 29 *Every ordinal is algebraic and the finite elements of a non-zero ordinal α are given by the set $\{\gamma + 1 \mid \gamma + 1 \in \alpha\} \cup \{0\}$.*

Proof: The fact that 0 is algebraic follows by a straightforward verification.

For a given non-zero ordinal α , we verify first that the set $\{\gamma + 1 \mid \gamma + 1 \in \alpha\} \cup \{0\}$ consists of finite elements.

The fact that 0 is finite is trivial since each ordinal contains the ordinal 0 as a subset.

Let $\beta \in \alpha$ be a successor ordinal, where say $\beta = \gamma + 1$. We show that β is finite. Assume that D is a (directed) subset of α and that $\beta = \gamma + 1 \subseteq \cup D$, where say $\cup D = \nu$. We obtain that $\gamma \in \cup D$, i.e. $\gamma \in \mu$ for some $\mu \in D$.

We distinguish two cases, depending on whether ν is a successor or a limit.

For the first case, we assume that ν is a successor ordinal, say $\rho + 1$. Then one can easily verify that $\nu \in D$. Indeed, otherwise if $\nu \notin D$, we obtain that $\forall \alpha \in D. \alpha \in \nu = \rho + 1$ and thus $\forall \alpha \in D. \alpha \subseteq \rho$ which implies that $\cup D \subseteq \rho$. Hence we have the contradiction that $\rho + 1 \subseteq \rho$. Hence $\nu \in D$.

We recall that $\gamma \in \mu \subseteq \nu = \rho + 1$. In particular we have that $\gamma \in \nu$ and thus $\gamma \subseteq \rho$. Hence $\beta = \gamma + 1 \subseteq \rho + 1 = \nu$, which implies that $\beta \subseteq \nu$, where $\nu \in D$. Hence $\gamma + 1$ is finite.

For the second case, we assume that ν is a limit ordinal. We recall that $\beta = \gamma + 1 \subseteq \cup D = \nu$. Since ν is by assumption a limit ordinal, we obtain that $\gamma + 1 \neq \nu$ and thus $\gamma + 1 \in \nu = \cup D$. Hence $\gamma + 1 \in \mu$ for some $\mu \in D$ and thus $\gamma + 1$ is finite.

To show the converse, we need to verify that every finite element of α is a successor ordinal or 0. Since we have remarked that 0 is a finite element, it suffices to show that every finite element e of α which is not 0, is a successor ordinal. We assume by way of contradiction that e is a finite element which is a non-zero limit ordinal. In that case we obtain that $e = \cup e$. It is straightforward to verify that this fact, combined with the fact that $e \neq 0$, implies that e is not finite.

Finally, we need to show that every ordinal α is algebraic. For this we need to show that each element β of α is the supremum of a subset of the set $\{\gamma + 1 \mid \gamma + 1 \in \alpha\} \cup \{0\}$. Let β be an element of α . In case β is a successor ordinal, the result follows trivially. So we can assume that β is a limit ordinal. In that case we have that $\beta = \cup \beta$. Clearly if $\beta = 0$ then $\beta = \cup 0$ and thus we can assume that $\beta \neq 0$.

Since every limit ordinal $\mu \in \beta$ is such that $\mu + 1 \in \beta$, we obtain that $\beta = \cup \{\nu + 1 \mid \nu \in \beta\}$. In other words, each limit ordinal β is the supremum of the successor ordinals below β and thus β is the supremum of the set of finite elements below β .

Hence we have shown that each ordinal is algebraic.

□

Lemma 30 ω_1 is the first ordinal which is not ω -algebraic.

Proof: To verify that any ordinal α strictly smaller than ω_1 is ω -algebraic, it suffices, by Lemma 29, to verify that the set of the finite elements $F(\alpha)$ is countable. This last fact however follows since any ordinal α strictly below ω_1 is countable.

Next, we verify that the ordinal ω_1 is not ω -algebraic. Readers skilled in set theory will notice this follows immediately from the fact that ω_1 is a regular cardinal (cf. [HJ84], Chapter 10, Theorem 2.3). For the sake of completeness we provide an alternative argument. By Lemma 29, the set of finite elements $F(\omega_1)$ is the set of the successor ordinals of ω_1 supplemented by the element 0. Since ω_1 is a limit ordinal, we obtain for each of its elements α which is a limit ordinal, that $\alpha + 1$ belongs to ω_1 . Hence the set of limit ordinals of ω_1 has a cardinality below the cardinality of the set of successor ordinals of ω_1 . So if ω_1 would have countably many finite elements then ω_1 would be countable. Hence ω_1 is not ω -algebraic. □

Definition 31 An algebraic partial order is sequentially adequate if every element of this partial order is the supremum of an eventually increasing sequence of finite elements.

Lemma 32 Every ω -algebraic partial order is sequentially adequate.

Proof: We present a sketch. The argument is similar to the one of [DP90], exercise 3.5 of Chapter 3.

Given an ω -algebraic partial order (P, \sqsubseteq) . Let x be any element of P and let D be a directed subset of finite elements of P such that $x = \sqcup D$. Since $F(P)$ is countable, we obtain that D is countable, say $D = \{x_0, x_1, \dots, x_n, \dots\}$. For each finite subset F of D , let u_D be an upper bound of F .

Inductively define subsets D_i of D as follows:

$$D_0 = \{x_0\} \text{ and } D_{i+1} = D_i \cup \{y_{i+1}, u_{D_i \cup \{y_{i+1}\}}\},$$

where y_{i+1} is the element x_n in $D - D_i$ with subscript n chosen as small as possible. One can verify that the sequence $(u_i)_i$, defined by $\forall i \geq 0. u_i = u_{D_i \cup \{y_{i+1}\}}$, is an increasing sequence in D with supremum $\sqcup D$. □

Comment: The above result implies that the notions of completion of a partially ordered set with respect to sequential convergence and with respect to convergence of countable directed sets, coincide. On the other hand, it follows immediately from

Markowsky's generalization ([Mar76]) of Iwamura's lemma ([Iwa44]), that the notions of completeness with respect to these two kinds of convergence also coincide.

The following lemma implies that the converse of Lemma 32 does not hold, since we obtain that the ordinal ω_1 is a sequentially adequate algebraic partial order which is not ω -algebraic.

Lemma 33 *$\omega_1 + 1$ is the first ordinal which is not sequentially adequate.*

Proof: To show that $\omega_1 + 1$ is not sequentially adequate, we argue by contradiction.

If $\omega_1 + 1$ were sequentially adequate, then in particular its maximum ω_1 would be the supremum of a sequence of finite elements from $\omega_1 + 1$. Without loss of generality we can assume that this sequence does not contain the ordinal 0. Hence, by Lemma 29, the sequence consists of successor ordinals. Since ω_1 is a limit ordinal, it does not belong to this sequence and hence all elements from the sequence are strictly below ω_1 . Since ω_1 is the first uncountable ordinal, each of these finite elements is countable. However, since the union of countably many countable sets is countable, we obtain the contradiction that ω_1 is countable. Hence $\omega_1 + 1$ is not ω -algebraic.

The verification that every ordinal strictly below $\omega_1 + 1$ is sequentially adequate proceeds in two steps.

First we remark that, by Lemma 30, any ordinal strictly below ω_1 is ω -algebraic and hence sequentially adequate.

Next we verify that ω_1 is sequentially adequate. Consider any element $\beta \in \omega_1$. We remark that β is countable. If β is a successor ordinal, it follows that β is a finite element and hence trivially is the limit of a countable sequence of finite elements. In case β is a limit ordinal, we obtain, via an argument similar to the end of the proof of Lemma 29, that β is the limit of the successor ordinals below β . Since β is countable, this implies that β is the limit of a countable sequence of finite elements.

□

We state a lemma characterizing the ideal completion of an ordinal.

Lemma 34 *The ideal completion of an ordinal α is its successor ordinal $\alpha + 1$.*

Proof: The ideals of an ordinal α are its downwardly closed directed subsets. It is easy to verify that these are precisely the ordinals less than or equal to α . Hence it is easy to see that the ideal completion of α is $\alpha + 1$.

□

Corollary 35 *The ideal completion is not sequentially adequate.*

Proof: We present a sketch. Consider the ordinal ω_1 which has $\omega_1 + 1$ as its ideal completion. We remark that $\omega_1 + 1$ is not sequentially adequate by Lemma 33. From the proof of Lemma 33, we know that there is no sequence of finite elements which converges to ω_1 ; that is there is no sequence of successor ordinals which converges to ω_1 . Hence, by an argument similar to the one at the end of the proof of Lemma 29, there is no sequence of ordinals strictly below ω_1 and converging to ω_1 .

So we have shown that there is no sequential completion which can replace the general ideal completion.

□

We have remarked in Subsection 4.1 that the Yoneda completion reduces to the ideal completion for the case of partial orders. Hence, by Corollary 35, we obtain that in general the Yoneda completion is not sequentially adequate. In particular we obtain that the sequential Yoneda completion of [BvBR98] is not sequentially adequate for the Yoneda completion.

Similarly, we obtain that the Smyth completion is not sequentially adequate.

We discuss in the following some implications of sequential inadequacy for the case of the Yoneda completion.

We recall that Theorem 5.4 of [BvBR98] states that the sequential Yoneda completion of a quasi-metric space always is an *algebraic* complete quasi-metric space. We recall that the use of the term algebraic in [BvBR98] is different from our own (cf. the remark preceding Definition 17). However, in the context of quasi-metrics encoding an algebraic partial order, the terminology “algebraic” of [BvBR98] is equivalent to our notion of sequential adequacy.

Hence, according to [BvBR98], the sequential Yoneda completion of a quasi-metric space which encodes a partial order, is always sequentially adequate, since the sequential Yoneda completion is a quasi-metric space encoding an algebraic partial order. This is true in the context of [BvBR98] as only *sequential* completions are considered.

However, if we consider the Yoneda completion defined in Subsection 4.1, it is clear this result can no longer hold. This follows from the fact that the Yoneda completion is not sequentially adequate as remarked above. However, one can show that the Yoneda completion of a quasi-metric space is algebraic (Corollary 23).

A related problem arises for the open problem on the idempotency as stated in [BvBR98]. We show that the idempotency characterization obtained in Theorem 26 cannot be valid in general in the context of the *sequential* Yoneda completion.

Clearly, the completion presented in [BvBR98] is idempotent on the quasi-metric space which encodes ω_1 , since increasing sequences of elements from ω_1 necessarily have a supremum which again belongs to ω_1 . However the quasi-metric space encoding ω_1 is not Smyth-completable, since it allows for strictly increasing sequences and

thus for Cauchy sequences which are not biCauchy.

The problems clearly arise from the fact that the sequential Yoneda completion cannot capture the general ideal completion, for the case of quasi-metric spaces encoding a partial order.

We conclude the section by discussing two examples for which the sequential adequacy of the Yoneda completion can be guaranteed and thus for which the above problems do not arise.

A first example is the class consisting of those quasi-metric spaces encoding a partial order for which the Yoneda completion yields an ω -algebraic partial order. The fact that the Yoneda completion is sequentially adequate on this class follows from Lemma 32 combined with the fact that for the case of partial orders, the Yoneda completion reduces to the ideal completion.

In particular we obtain that the Yoneda completion is sequentially adequate on the class of quasi-metric spaces encoding a countable partial order. Indeed, it is easy to verify that the ideal completion of a countable partial order is ω -algebraic. Hence the Yoneda completion of a quasi-metric space encoding such a partial order is ω -algebraic.

A second example is the class of Smyth-completable quasi-metric spaces. The Yoneda completion is sequentially adequate on this class since, by Theorem 26, the Yoneda completion of a Smyth-completable quasi-metric space coincides with the bicompletion and it is well known that the bicompletion of a quasi-metric space is sequentially adequate (cf. [DiC71]).

6 Preservation of topological properties

We recall a result which will be used frequently in the proofs. Every quasi-uniform space (X, \mathcal{U}) is densely embedded in its Smyth completion $(\widetilde{X}, \widetilde{\mathcal{U}}, \widetilde{\mathcal{T}})$, with respect to the conjugate topology $\mathcal{T}(\widetilde{\mathcal{U}}^{-1})$ (cf. a result of Sünderhauf discussed in [Kün93, p. 322]).

By Proposition 19, we immediately obtain that each quasi-metric space is densely embedded in its Yoneda completion with respect to the conjugate topology.

Theorem 36 *Let (X, d) be a quasi-metric space.*

- (1) *If (X, d) is totally bounded then $(\overline{X}^Y, (\overline{d}^Y)^*)$ is compact.*
- (2) *(X, d) is precompact if and only if $(\overline{X}^Y, \overline{d}^Y)$ is precompact.*
- (3) *If (X, d) is compact then $(\overline{X}^Y, \overline{d}^Y)$ is compact.*

Proof: (1) We remark that if a quasi-metric space (X, d) is totally bounded then it is Smyth-completable ([Sün93]) and thus the Yoneda completion of the space coincides with the bicompletion. The result follows since the bicompletion of a totally bounded

space (X, d) is compact with respect to the topology $\mathcal{T}(\bar{d}^Y)^*$ (e.g. [FL82], Proposition 3.36).

To show (2), we recall a result from [KL95]:

Proposition 4 (c): Let D be a $\mathcal{T}(\mathcal{U}^{-1})$ -dense subspace of a quasi-uniform space (X, \mathcal{U}) . Then $(D, \mathcal{U}|_D)$ is precompact if and only if (X, \mathcal{U}) is precompact.

So (2) follows immediately from Proposition 4 (c) of [KL95] and by the fact that a quasi-metric space is densely embedded in its Yoneda completion with respect to the conjugate topology.

To show (3), we remark that in a quasi-metric space, each point has a neighborhood base consisting of sets which are closed in the conjugate topology (e.g. [Kel63]). Let \mathcal{G} be an open cover of (\bar{X}^Y, \bar{d}^Y) . Then consider those open sets of this space whose closure with respect to the conjugate topology is contained in some member of the cover \mathcal{G} (i.e. use pairwise regularity). This new cover has a finite subcollection covering $\{\bar{x} : x \in X\}$ by our assumption. The union of the closures with respect to the conjugate topology of this subcollection covers \bar{X}^Y , since the set $\{\bar{x} : x \in X\}$ is densely embedded in \bar{X}^Y with respect to the conjugate topology. We conclude that \mathcal{G} has a finite subcover covering \bar{X}^Y .

□

Corollary 37 *The Yoneda completion preserves total boundedness and compactness with respect to the associated symmetric topology.*

Proof: Totally bounded spaces are Smyth-completable ([Sün95]) and total boundedness is preserved by bicompletions ([FL82]). Also, a quasi-metric space which is compact with respect to its associated symmetric topology is totally bounded and hence its Yoneda completion is compact with respect to the associated symmetric topology.

□

We remark that the results stated in the corollary as well as in part (1) of Theorem 36 are not new in the sense that the Yoneda completion for these cases is the bicompletion and hence the results are implied by well-known results regarding this last completion. However parts (2) and (3) of Theorem 36 truly involve new results concerning the Yoneda completion.

For the Smyth completion, analogous results hold.

Theorem 38 *Let (X, d) be a quasi-metric space.*

- (1) *If (X, d) is totally bounded then $(\bar{X}, (\bar{\mathcal{U}}_d)^*)$ is compact.*
- (2) *(X, d) is precompact if and only if (X, \mathcal{U}_d) is precompact.*
- (3) *If (X, d) is compact then the Smyth completion $(\bar{X}, \bar{\mathcal{U}}_d, \bar{\mathcal{T}})$ of the topological quasi-uniform space $(X, \mathcal{U}_d, \mathcal{T}(\mathcal{U}_d))$ is compact with respect to $\mathcal{T}(\bar{\mathcal{U}}_d)$.*

Proof: We remark that (1) and (2) follow from (1) and (2) of Theorem 36, combined with Proposition 19.

The proof of (3) can be obtained as for (3) of Theorem 36, using the fact that by Proposition 10 of [Kün93], each point of a topological quasi-uniform space has a neighborhood base consisting of sets which are closed with respect to the conjugate quasi-uniformity and the above cited result on the dense embedding of (X, \mathcal{U}) in its Smyth completion $(\widetilde{X}, \widetilde{\mathcal{U}}, \widetilde{\mathcal{T}})$, with respect to the conjugate topology. □

Corollary 39 *The Smyth completion preserves total boundedness and compactness with respect to the associated symmetric topology.* □

Again, only part (2) and (3) of Theorem 38 are new results.

We remark that hereditary precompactness is *not* preserved by the Yoneda completion and thus (by Proposition 19) is not preserved by the Smyth completion.

Counterexample 1: Let $X = \{(n, m) \in \omega \times \omega : n \leq m\}$. Define a partial order on X by $(x, y) \leq (x', y')$ if (1) $x = x'$ and $y \leq y'$ or (2) $y < x'$.

Then (X, \leq) is clearly hereditarily precompact (cf. also [KR95]); of course we consider the quasi-metric d such that $d(x, y) = 0$ if $x \leq y$.

For each $k \in \omega$, set $(a_k) = ((k, l+k))_{l \in \omega}$. Note that each (a_k) is a Cauchy sequence in X . But observe that $\vec{d}^y((a_p), (a_r)) = 1$ whenever $p, r \in \omega$ and $p < r$. To this end choose $k \in \omega$ such that $k > r$. Then $(p, p+k) \leq (r, r+l)$ is not satisfied whenever $l \in \omega$. Hence we have found a subspace that is not precompact.

We conclude that $(\overline{X}^Y, \vec{d}^Y)$ is not hereditarily precompact.

On the other hand, if the conjugate of a quasi-pseudo-metric space is hereditarily precompact, then the conjugate of the Yoneda completion is hereditarily precompact. The same holds for the Smyth completion.

This follows since, if the conjugate is hereditarily precompact, then by Example 6 of [Kün93], the space is Smyth-completable. Furthermore the bicompletion of a hereditarily precompact space is hereditarily precompact ([KL95]).

We remark that counterexample 1 can be adapted in order to show that precompactness of a quasi-metric space does not imply compactness of its Yoneda completion. Again by Proposition 21, the same holds for the quasi-uniform topology of the Smyth completion. On the other hand it follows from Corollary 41 below that the topology of the Smyth completion is compact.

Counterexample 2: Let (X, \leq) be the partially ordered space defined in counterexample 1. Set $p((x, y), (x', y')) = 0$ if $x = x'$ and $y \leq y'$, $p((x, y), (x', y')) = 2^{-n}$ if $(x, y) \leq (x', y')$, $x < x'$ and $x = n$ and $p((x, y), (x', y')) = 1$ otherwise.

One can easily check that (X, p) is a hereditarily precompact quasi-metric space. We omit the straightforward verifications. Consider for each $n \in \omega$ the sequence $a_s = (s, s + k)_{k \in \omega}$. Clearly each such sequence is a Cauchy sequence. Suppose that the sequence $(a_s)_{s \in \omega}$ has a cluster point $(x_n)_n$ in \overline{X}^y . Then $(x_n)_n$ cannot be unbounded in the first coordinate by the definition of \overline{p}^y , because each a_s is bounded in the first coordinate so that $\overline{p}^y((x_n)_n, a_s) < 1$ would be impossible. But if $(x_n)_n$ is bounded in the first coordinate, then it cannot be arbitrarily close to $(a_s)_s$ for large s by the definition of p . We conclude that the Yoneda completion of (X, p) is not compact.

The results on the preservation of topological properties by the two types of completion are summarized in the table below. The following abbreviations are used: TB for “totally bounded”, C for “compact”, PC for “precompact” and HPC for “hereditarily precompact”. An indexing of any of the abbreviations with a star, indicates that the property holds with respect to the associated symmetric topology. For instance C^* , for a given quasi-metric space (X, d) , indicates that the metric space $(X, \mathcal{T}(d^*))$ is compact.

An implication of the form “ $P \Rightarrow Q$ ” indicates that in case the property P holds for a given quasi-metric space, the property Q holds for both the Yoneda completion and for the Smyth completion of this space. A crossed out implication of the form “ $P \not\Rightarrow Q$ ” indicates that the property P for a given quasi-metric space does not imply property Q for its Yoneda completion nor for its Smyth completion.

Preservation of topological properties		
$TB \Rightarrow C^*$	$C \Rightarrow C$	$PC \Rightarrow PC$
$TB^* \Rightarrow TB^*$	$C^* \Rightarrow C^*$	$HPC \not\Rightarrow HPC$

Finally, we show that for the Smyth completion, precompactness of a T_0 topological quasi-uniform space $(X, \mathcal{U}, \mathcal{T})$ implies compactness of the space $(\widetilde{X}, \widetilde{\mathcal{T}})$ (cf. Corollary 41). We remark that we do not obtain this result for the topology $\mathcal{T}(\widetilde{\mathcal{U}})$ as remarked in the paragraph preceding counterexample 2.

Proposition 40 *Let $(X, \mathcal{U}, \mathcal{T})$ be a T_0 Smyth complete topological quasi-uniform space such that (X, \mathcal{U}) is precompact. Then (X, \mathcal{T}) is compact.*

Proof: Let \mathcal{G} be an ultrafilter on X . Set $M(U) = \{x \in X \mid U(x) \in \mathcal{G}\}$ whenever $U \in \mathcal{U}$. By [Kün93, Proof of Proposition 13] there is a left K -Cauchy filter \mathcal{H} on (X, \mathcal{U}) such that $\{M(U) \mid U \in \mathcal{U}\} \subseteq \mathcal{H}$. Let \mathcal{E} be the filter on X generated by $\{U(H) \mid U \in \mathcal{U}, H \in \mathcal{H}\}$. Clearly \mathcal{E} is a round S-Cauchy filter on the topological quasi-uniform space $(X, \mathcal{U}, \mathcal{T}(\mathcal{U}))$. Consider the identity TQUS-morphism $i : (X, \mathcal{U}, \mathcal{T}(\mathcal{U})) \rightarrow (X, \mathcal{U}, \mathcal{T})$. For its extension \widetilde{i} to the completion of $(X, \mathcal{U}, \mathcal{T}(\mathcal{U}))$ we have by [Sün93, Lemma 14] $\mathcal{N}(\widetilde{i}(\mathcal{E})) = \{A \subseteq X : \text{There are } E \in \mathcal{T} \text{ and } U \in \mathcal{U} \text{ such that } E \in \mathcal{E}, E <_U A\}$, where $<_U$ refers to the topological quasi-uniform space $(X, \mathcal{U}, \mathcal{T})$. Clearly $\mathcal{N}(\widetilde{i}(\mathcal{E})) \subseteq \mathcal{E} \subseteq \mathcal{H}$.

Let $O \in \mathcal{N}(\widetilde{i}(\mathcal{E}))$. There are $E \in \mathcal{E}$ and $U \in \mathcal{U}$ such that $U(E) \subseteq O$. Since $M(U) \cap E \in \mathcal{H}$, we can choose $y \in M(U) \cap E$. Then $U(y) \in \mathcal{G}$ and $U(y) \subseteq U(E) \subseteq$

O. We conclude that \mathcal{G} converges with respect to the topology \mathcal{T} . Hence (X, \mathcal{T}) is compact. □

Corollary 41 *Let $(X, \mathcal{U}, \mathcal{T})$ be a T_0 precompact topological quasi-uniform space and let $(\tilde{X}, \tilde{\mathcal{U}}, \tilde{\mathcal{T}})$ be its Smyth completion. Then $(\tilde{X}, \tilde{\mathcal{T}})$ is compact.*

Proof: We know that by a result of Sünderhauf [Kün93, p. 322], which has already been used several times before, X is $\mathcal{T}(\tilde{\mathcal{U}}^{-1})$ -dense in \tilde{X} . Thus $(\tilde{X}, \tilde{\mathcal{U}})$ is precompact by [KL95, Prop. 4(c)]. The assertion follows from the preceding result. □

Observe that the fact that hereditary precompactness is preserved by the classical bicompletion ([KL95]), but not by the Yoneda completion nor by the Smyth completion, ought not necessarily to be viewed as a defect of these last two completions.

We recall ([KR]) that a hereditarily precompact quasi-uniform space which encodes a partial order, encodes a well-quasi-order. As mentioned in the introduction, the theory of well-quasi-orders is fundamental in termination proofs for rewrite systems. The strategy involves an introduction of a well-quasi-order on the terms, where the rewrite rules are shown to lead to strict decreases in the ordering (e.g. [DJ90]).

We remark that semantics for rewrite systems typically involve free (universal algebra type) models. In general, when solutions for fixed point equations are considered or when an equational theory is extended by a recursion operator, a model for the original theory can be extended to a model for the extended theory through a partial order completion (e.g. [CV74] and [CV76], where recursive equations are considered in combination with rewrite rules).

One would intuitively expect the introduction of a recursion operator and a corresponding extension of the term model via a partial order completion to violate termination and hence in particular to violate the well-quasi-ordering under consideration. The negative results regarding hereditary precompactness for the Yoneda completion and Smyth completion can be viewed as a (formal) reflection of the above intuition.

The fact that the bicompletion preserves hereditary precompactness leads to some speculation regarding applications of the above kind, in the context of Smyth-completable quasi-metric spaces.

We remark that for Smyth-completable spaces which encode a partial order, every Cauchy sequence is eventually constant. In theories where decidability is an issue, as in [Sta86] and possibly in the related [CV74] and [CV76], such models may be of (speculative) use. In [Sta86], an equational theory $\lambda\sqcup$ of typed lambda terms is considered, aimed at the analysis of fixed point equations. It is shown in particular

(corollary 1 of [Sta86]) that a fixed point equation $Tx = x$ is solvable in every model of $\lambda\sqcup$ if and only if for some n $\lambda\sqcup \vdash T^{n+1} \perp = T^n \perp$; that is in the term model, the chain of iterations $(T^n \perp)_n$ is eventually constant.

7 Conclusion

We have obtained a generalized Yoneda completion in terms of nets and have shown that this completion is not sequentially adequate in general.

A solution has been presented to the open question raised in [BvBR98], regarding the class of spaces on which the Yoneda completion is idempotent.

We have characterized the largest class of quasi-metric spaces on which the Yoneda completion is idempotent to consist of the Smyth-completable quasi-metric spaces. On this class, which includes the totally bounded spaces, the Yoneda completion and the Smyth completion reduce to the bicompletion. As such both completions have been shown to preserve total boundedness and compactness with respect to the symmetric associated topology.

The fact that both completions coincide in the context of the theory of totally bounded spaces illustrates that this theory poses no difficulty and hardly any possibility of controversy ([Smy91]).

It also entails that the Yoneda completion and the Smyth completion perform equally well on the class of totally bounded spaces and indeed can be viewed as alternative extensions of the bicompletion to the context of non-Smyth-completable spaces.

The fact remains that the Yoneda completion is not idempotent in general. We have shown however that despite this fact, the topological properties of precompactness and compactness are preserved by the Yoneda completion as well as by the Smyth completion. However not all properties which for a classical completion, as the bicompletion, might be expected to be preserved, are preserved by the Yoneda completion nor by the Smyth completion. This has been shown by a counterexample for the hereditary precompactness property. As indicated above, this negative result ought to be expected since the well-quasi-ordering (hereditary precompactness) of a term-model for a rewrite system will in general be violated by a partial order completion. The negative result does indicate however that some caution is needed in handling these non-standard completions.

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