Quasi-metric Properties of Complexity Spaces*

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Abstract

The complexity (quasi-metric) space has been introduced as a part of the development of a topological foundation for the complexity analysis of algorithms ([12]). Applications of this theory to the complexity analysis of Divide & Conquer algorithms have been discussed in [12].

Here we obtain several quasi-metric properties of the complexity space. The main results obtained are the Smyth-completeness of the complexity space and the compactness of closed complexity spaces which possess a (complexity) lower bound. Finally, some implications of these results in connection to the above mentioned complexity analysis techniques are discussed and the total boundedness of complexity spaces with a lower bound is discussed in the light of Smyth's computational interpretation of this property ([14]).

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1 Introduction

The letters $\mathcal{N}$, ω, $\mathcal{R}$ and $\mathcal{R}^+$ denote the set of positive integers, of nonnegative integers, of real numbers and of nonnegative real numbers, respectively.

In this paper a quasi-metric on a set X is a nonnegative real-valued function d on $X \times X$ such that for all $x, y, z \in X$: (1) $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$ and (2) $d(x, z) \leq d(x, y) + d(y, z)$.

The function u on $\mathcal{R} \times \mathcal{R}$, where $u(x, y) = (y - x) \vee 0$, is an example of a quasi-metric.

A quasi-metric space is a pair (X, d) consisting of a set X and of a quasi-metric d on X.

If d is a quasi-metric on X, then the function d^{-1}, defined on $X \times X$ by $d^{-1}(x, y) = d(y, x)$, is also a quasi-metric on X, called the conjugate of d. By d^s we denote the metric defined on X by $d^s(x, y) = d(x, y) \vee d^{-1}(x, y)$ for all $x, y \in X$.

Each quasi-metric d on X generates a T_0 topology $T(d)$ on X which has as a base the family of balls $\{B_d(x, r) | \, x \in X, r > 0\}$, where $B_d(x, r) = \{y \in X \mid d(x, y) < r\}$.

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Our basic references for quasi-uniform spaces are [2] and [5].

Each quasi-metric on X induces a quasi-uniformity $\mathcal{U}_d$ on X which has a a base the family of sets of the form $\{(x, y) \in X \times X \mid d(x, y) < 2^{-n}\}$, where $n \in \mathcal{N}$ (e.g. [2, p. 3]).

A quasi-uniform space $(X,\mathcal{U})$ is called *bicomplete* if the uniform space $(X,\mathcal{U}^s)$ is complete (where $\mathcal{U}^s = \mathcal{U} \vee \mathcal{U}^{-1}$).

A filter $\mathcal{F}$ on a quasi-uniform space $(X,\mathcal{U})$ is called *left K-Cauchy* (e.g. [9]) if for each $U \in \mathcal{U}$ there is an $F \in \mathcal{F}$ such that $U[x] \in \mathcal{F}$ for all $x \in F$.

We refer the reader to [15] and [16] for an introduction to the Smyth-completion. We recall some results by Künzi on the Smyth-completion ([4]). A quasi-uniform space is Smyth-completable if and only if every left K-Cauchy filter on $(X,\mathcal{U})$ is a Cauchy filter on the uniform space $(X,\mathcal{U}^s)$. A quasi-uniform space is Smyth-complete if and only if every left K-Cauchy filter on $(X,\mathcal{U})$ converges to a unique point in $(X,\mathcal{U}^s)$. Thus every T_0 Smyth-completable bicomplete quasi-uniform space is Smyth-complete.

A quasi-metric space (X, d) is called *Smyth-completable* (respectively *Smyth-complete*, *bicomplete*) if the quasi-uniform space $(X,\mathcal{U}_d)$ is Smyth-completable (respectively Smyth-complete, bicomplete).

We recall that the generalized metric spaces known as the "weightable quasi-metric spaces" have been introduced by Matthews in [7] as a part of the study of the denotational semantics of dataflow networks. A quasi-metric space (X, d) is *weightable* if there exists a function $w: X \to \mathcal{R}^+$ such that for every $x, y \in X$, $d(x, y) + w(x) = d(y, x) + w(y)$. The function w is called a *weighting function*, $w(x)$ is the *weight* of x and the quasi-metric d is *weightable by the function* w.

The complexity (quasi-metric) space was introduced in [12] as a part of the development of a topological foundation for the complexity analysis of algorithms (see also [13]). Via the analysis of its dual, we here show that the complexity space possesses several quasi-metric properties which are interesting from a Computer Science point of view (in the context of the Smyth-completion). We prove that the (dual) complexity space is a Smyth-complete Baire quasi-metric space and that the closed complexity spaces which possess a (complexity) lower bound are compact. Furthermore, some implications of these results in connection to the above mentioned complexity analysis techniques are discussed and the total boundedness of complexity spaces with a lower bound is discussed in the light of Smyth's computational interpretation of this property ([14]).

2 The dual of the complexity space

As mentioned above, the main object of our study is the complexity space (C, d_C), where

$$C = \{f: \omega \to (0, +\infty] \mid \sum_{n=0}^{\infty} 2^{-n} \frac{1}{f(n)} < +\infty\}$$

and d_C is the quasi-metric on C defined by

$$d_C(f,g) = \sum_{n=0}^{\infty} 2^{-n}[(\frac{1}{g(n)} - \frac{1}{f(n)}) \vee 0]$$

whenever $f, g \in C$. Any subspace of (C, d_C) is also called a complexity space (cf. [12]).

We define the quasi-metric space (C^*, d_{C^*}) as follows:

$$C^* = \{f\colon \omega \to \mathcal{R}^+ \mid \sum_{n=0}^{\infty} 2^{-n} f(n) < +\infty\}$$

and d_{C^*} is the quasi-metric on C^* defined by

$$d_{C^*}(f,g) = \sum_{n=0}^{\infty} 2^{-n}[(g(n) - f(n)) \vee 0]$$

whenever $f, g \in C^*$.

Note that the inversion function $\Psi\colon C^* \to C$ is an isometry from (C^*, d_{C^*}) to (C, d_C) since $d_C(\Psi(f), \Psi(g)) = d_C(\frac{1}{f}, \frac{1}{g}) = d_{C^*}(f,g)$ whenever $f, g \in C^*$. Hence, from now on, we shall refer to the space (C^*, d_{C^*}) as the *dual complexity space*. Any subspace of (C^*, d_{C^*}) is also called a dual complexity space.

The fact that the dual space (C^*, d_{C^*}) admits a structure of semilinear quasi-normed space, in the sense of [1] and [10], provides a first motivation for the use of the dual complexity space rather than the original one in this context. As such a construction is not necessary for our purposes here, we will discuss this approach elsewhere ([11]).

A second motivation for the use of the dual space instead of the original space is the fact that the definition of the dual is mathematically somewhat more appealing, since d_{C^*} is derived from the restriction to $\mathcal{R}^+$ of the quasi-metric u, defined in Section 1, for which u^s is the standard metric. Consequently, the presentation of the proofs becomes somewhat more elegant.

We will illustrate in Section 4 that it is still possible to carry out the complexity analysis of algorithms based on the dual complexity space. Since the intuitive interpretation of the original functional will become less straightforward in such an approach, the original complexity space remains an appropriate tool to investigate the complexity of algorithms, where, as mentioned above, the mathematical elegance of the dual makes it an appropriate tool for a topological study of these spaces. Still, even from a Computer Science point of view, the dual has a definite appeal, since in this context, the complexity space has a minimum $\perp$ which corresponds directly to the minimum of semantic domains.

We recall from [12] that the complexity space (C, d_C) is a weightable quasi-metric space with weighting function w_C defined on C by $w_C(f) = \sum_{n=0}^{\infty} 2^{-n} f(n)$ whenever $f \in C$. From this result and the duality discussed above, via the isometry Ψ, we obtain the following proposition.

Proposition 1 *The dual complexity space* (C^*, d_{C^*}) *is a weightable quasi-metric space with weighting function* w_{C^*} *defined on* C^* *by* $w_{C^*}(f) = \sum_{n=0}^{\infty} 2^{-n} f(n)$ *whenever* $f \in C^*$.

Corollary 2 *The dual complexity space* (C^*, d_{C^*}) *is a Smyth-completable quasi-metric space.*

Proof: It is shown in [4, Proposition 15] that every weightable quasi-metric space is Smyth completable. □

It is well known that, similarly to the metric case, the function d_ρ defined on $\mathcal{R}^\omega \times \mathcal{R}^\omega$ by $d_\rho(x, y) = \sum_{n=0}^{\infty} 2^{-n}[u(x(n), y(n)) \wedge 1]$ is a quasi-metric on $\mathcal{R}^\omega$ such that the topology $\mathcal{T}(d_\rho)$ on $\mathcal{R}^\omega$ induced by d_ρ coincides with the topology of the product space $\prod_{n\in\omega}(\mathcal{R}, u)$.

Clearly, $d_{C^*} \geq d_\rho$ on C^*. Furthermore, the following counterexample shows that the topology $\mathcal{T}(d_{C^*})$ is strictly finer than $\mathcal{T}(d_\rho)$. (For the sake of brevity, the metrics $(d_\rho)^s$ and $(d_{C^*})^s$ will be simply denoted by d_ρ^s and $d_{C^*}^s$, respectively).

Example 1: For each $k \in \omega$ we define a function $f_k \colon \omega \to \mathcal{R}^+$ by $f_k(n) = 0$ if $n < k$ and $f_k(n) = 2^k$ if $n \geq k$. Then $f_k \in C^*$ for all $k \in \omega$. Clearly $d_\rho^s(0, f_k) \to 0$. However the function $0 \in C^*$ is not a $\mathcal{T}(d_{C^*})$-cluster point of the sequence $(f_k)_{k\in\omega}$ since $\sum_{n=0}^{\infty} 2^{-n} f_k(n) = 2^k \sum_{n=k}^{\infty} 2^{-n} = 2$ for all $k \in \omega$.

On the other hand, since $(\mathcal{R}^+, u)$ is a weightable bicomplete quasi-metric space [4], [6], it is Smyth-complete, and thus, by [17, Corollary 6], $((\mathcal{R}^+)^\omega, d_\rho)$ is Smyth-complete. Our next example shows that the restriction of d_ρ to C^* is however not a Smyth-complete quasi-metric. Actually, we show that the space (C^*, d_ρ) is not bicomplete.

Example 2: For each $k \in \mathcal{N}$, define the function $f_k \colon \omega \to \mathcal{R}^+$ where $f_k(n) = 2^n - \frac{1}{2^k}$ if $n \leq k$ and $f_k(n) = 0$ if $n > k$. Clearly, we have that $f_k \in C^*$ for all $k \in \mathcal{N}$. An easy computation shows that for each $k \in \mathcal{N}$, $\sum_{n=0}^{\infty} 2^{-n}[|f_{k+1}(n) - f_k(n)| \wedge 1] = (\frac{1}{2^k} - \frac{1}{2^{k+1}}) \sum_{n=0}^{k} 2^{-n} + 2^{-(k+1)} < \frac{1}{2^{k+1}} \cdot 2 + \frac{1}{2^{k+1}}$.

So $d_\rho^s(f_k, f_{k+1}) < \frac{3}{2^{k+1}}$ for all $k \in \mathcal{N}$. Thus $(f_k)_{k\in\mathcal{N}}$ is a Cauchy sequence in the metric space (C^*, d_ρ^s).

On the other hand, the sequence $(f_k)_{k\in\mathcal{N}}$ converges pointwisely, with respect to the Euclidean topology, to the function $g \colon \omega \to \mathcal{R}^+$ defined by $g(n) = 2^n$ for all $n \in \omega$. Thus the sequence $(f_k)_{k\in\mathcal{N}}$ converges to g in the metric space $((\mathcal{R}^+)^\omega, d_\rho^s)$. However $g \notin C^*$ because $\sum_{n=0}^{\infty} 2^{-n} g(n) = +\infty$. So (C^*, d_ρ) is not bicomplete.

Remark: It also follows from [17, Corollary 6] that the product quasi-uniform space $\prod_{n\in\omega}(\mathcal{R}^+, \mathcal{U}_u)$ is Smyth-complete. Note that C^* is not a closed subset in the product uniform space $\prod_{n\in\omega}(\mathcal{R}^+, \mathcal{U}_u^s)$, as the preceding example shows. Furthermore the sequence $(f_k)_{k\in\omega}$ constructed in Example 1 shows that the uniformity $\mathcal{U}^s_{d_{C^*}}$ on C^* is strictly finer than the restriction of $\prod_{n\in\omega}(\mathcal{R}^+,\mathcal{U}^s_u)$ to C^*.

However (the proof of) our next result shows that (C^*, d_{C^*}) is bicomplete.

Theorem 3 *The dual complexity space (C^*, d_{C^*}) is Smyth-complete.*

Proof: Since (C^*, d_{C^*}) is a weightable quasi-metric space and every bicomplete weightable quasi-metric space is Smyth-complete, it suffices to show that the dual complexity space (C^*, d_{C^*}) is bicomplete. Let $(f_k)_{k\in\mathcal{N}}$ be a Cauchy sequence in the metric space $(C^*, d^s_{C^*})$. Since $d_C \geq d_\rho$ on C^*, the sequence is also a Cauchy sequence in the metric space (C^*, d^s_ρ). So the sequence converges pointwisely with respect to the Euclidean topology, to a unique function $g\colon \omega \to \mathcal{R}^+$.

First we show that $g \in C^*$.

Assume the contrary. Then for each $j \in \mathcal{N}$ there is an $m_j \in \omega$ such that

$$(1)\ j < \sum_{n=0}^{m_j} 2^{-n} g(n).$$

Since $(f_k)_{k\in\mathcal{N}}$ is a Cauchy sequence in the metric space $(C^*, d^s_{C^*})$, there exists a natural number k_1 such that for each $k \geq k_1$

$$(2)\ \sum_{n=0}^{\infty} 2^{-n}|f_k(n) - f_{k_1}(n)| < 1.$$

Let $j \in \mathcal{N}$. By the pointwise convergence of $(f_k)_{k\in\mathcal{N}}$ to g, there exists a natural number $k_0 \geq k_1$ such that $|g(n) - f_{k_0}(n)| < 2^{-j}$ for $n = 0, 1, , \ldots, m_j$. Therefore

$$(3)\ \sum_{n=0}^{m_j} 2^{-n}|g(n) - f_{k_0}(n)| < 2^{-j}\sum_{n=0}^{m_j} 2^{-n} < 2^{-(j-1)}.$$

It follows from (1) and (3) that

$$(4)\ j < 2^{-(j-1)} + \sum_{n=0}^{m_j} 2^{-n} f_{k_0}(n).$$

Since, by (2),

$$(5)\ \sum_{n=0}^{m_j} 2^{-n} f_{k_0}(n) < 1 + \sum_{n=0}^{m_j} 2^{-n} f_{k_1}(n)$$

we deduce, using (4) and (5), that for each $j \in \mathcal{N}$, $j < 2 + \sum_{n=0}^{m_j} 2^{-n} f_{k_1}(n)$, which contradicts the fact that $f_{k_1} \in C^*$. So $g \in C^*$.

Next we show that $d^s_{C^*}(g, f_k) \to 0$. Let $j \in \mathcal{N}$. Then there exists a $k(j) \in \mathcal{N}$ such that for every $k, m \geq k(j)$

$$(6)\ \sum_{n=0}^{\infty} 2^{-n}|f_k(n) - f_m(n)| < 2^{-3j}.$$

Since $f_{k(j)} \in C^*$ and $g \in C^*$, there exists $n_0 \in \mathcal{N}$ (depending on j) such that

$$(7)\ n_0 \cdot 2^{-(n_0-1)} < 2^{-3j}, \sum_{n=n_0}^{\infty} 2^{-n} f_{k(j)}(n) < 2^{-3j} \text{ and } \sum_{n=n_0}^{\infty} 2^{-n} g(n) < 2^{-3j}.$$

So there exists a $k_j \geq k(j)$ such that for every $k, m \geq k_j$

$$(8)\ \sum_{n=0}^{\infty} 2^{-n} |f_k(n) - f_m(n)| < 2^{-n_0}.$$

Choose any $k \geq k_j$. Then for each $n \in \omega$, where $0 \leq n \leq n_0 - 1$ there is an $m_n \geq k$ such that $|g(n) - f_{m_n}(n)| < 2^{-n_0}$ and so, using (8), $|g(n) - f_k(n)| < 2^{-n_0} + 2^n \cdot 2^{-n_0}$ for $0 \leq n \leq n_0 - 1$. Thus

$$\sum_{n=0}^{n_0-1} 2^{-n} |g(n) - f_k(n)| < 2^{-n_0} \sum_{n=0}^{n_0-1} (2^{-n} + 1) < 2^{-n_0} \cdot 2n_0 < 2^{-3j}.$$

Moreover, by (6) and (7):

$$\sum_{n=n_0}^{\infty} 2^{-n} |g(n) - f_k(n)| \leq \sum_{n=n_0}^{\infty} 2^{-n} g(n) + \sum_{n=n_0}^{\infty} 2^{-n} f_k(n) < 2^{-3j} + \sum_{n=n_0}^{\infty} 2^{-n} f_{k(j)}(n) + 2^{-3j} < 3 \cdot 2^{-3j}.$$

Thus we have shown that, for each $j \in \mathcal{N}$, there is a $k_j \in \mathcal{N}$ such that for every $k \geq k_j$, $\sum_{n=0}^{\infty} 2^{-n} |g(n) - f_k(n)| < 4 \cdot 2^{-3j} \leq 2^{-j}$.

We conclude that $d^s_{C^*}(g, f_k) \to 0$. So the dual complexity space is a Smyth-complete quasi-metric space. □

Theorem 4 *The dual complexity space (C^*, d_{C^*}) is a Baire Space.*

Proof: Since $d_{C^*}(f, 0) = 0$ for all $f \in C^*$, the function with constant value 0 is in every open set of $\mathcal{T}(d_{C^*})$. Hence the space (C^*, d_{C^*}) is a Baire Space. □

Remark: It follows from Theorems 3 and 4 and by the existence of the isometry Ψ constructed above, that the complexity space (C, d_C) is a Smyth-complete Baire quasi-metric space.

3 Total boundedness and compactness

We recall the definitions of precompact and totally bounded quasi-metric spaces ([2]). The totally bounded spaces form an example of a class of Smyth-completable spaces ([17]) and their relevance to Computer Science and, in particular, to Complexity Theory, has been discussed in [14].

Definition 5 A quasi-metric space (X, d) is called *precompact* if for each $\varepsilon > 0$ there is a finite subset A of X such that, for every $x \in X$, there is some $a \in A$ for which

$d(a, x) < \varepsilon$. (X, d) is said to be *totally bounded* if the metric space (X, d^s) is precompact.

Note that total boundedness implies precompactness and that for metric spaces, that is, for the symmetric case, the notions of total boundedness and precompactness coincide. This is not necessarily true for the nonsymmetric case. A counterexample is, for instance, given by the space $(\mathcal{R}^+, u^{-1})$. The space is precompact since for each $x \geq 0$, $u^{-1}(0, x) = 0$ but it is not totally bounded since the associated metric $(u^{-1})^s$ is the standard metric on the nonnegative reals.

The above result for the symmetric case has been extended by Künzi ([4, Proposition 12]) to the context of the Smyth-completable spaces. We formulate this proposition in terms of quasi-metric spaces.

Proposition 6 (*[4]*) *Every hereditarily precompact Smyth-completable quasi-metric space is totally bounded.*

Definition 7 A complexity space $(\mathcal{F}, d_C)$, where $\mathcal{F} \subseteq C$, has a *lower bound* $m \in C$ if for each $f \in \mathcal{F}$ and each $n \in \omega$, $m(n) \leq f(n)$. A dual complexity space $(\mathcal{F}, d_{C^*})$ has an *upper bound* $m \in C^*$ if for each $f \in \mathcal{F}$ and each $n \in \omega$, $f(n) \leq m(n)$.

We show that complexity spaces are totally bounded under the assumption that they have a lower bound.

This assumption can be motivated as follows. In general, for a given problem, a minimum amount of complexity will be required in order for any program to carry out the prescribed task. That is, for a class of programs calculating a given partial recursive function, there will typically exist a lower bound on the complexity of the programs. Hence many complexity arguments involve classes of complexity functions which have a complexity lower bound (e.g. [3], [8]), which makes the study of complexity spaces with a lower bound a worthwhile goal.

To obtain our result it suffices to show the following dual version.

Theorem 8 *A dual complexity space* $(\mathcal{F}, d_{C^*})$ *with an upper bound is totally bounded.*

Proof: Let $(\mathcal{F}, d_{C^*})$ be a dual complexity space with an upper bound, say $m \in C^*$.

Since weightedness is a hereditary property and since the dual complexity space (C^*, d_{C^*}) is weightable, we obtain that the dual complexity space $(\mathcal{F}, d_{C^*})$ is a weightable space. Hence this space is Smyth-completable and thus it suffices, by Proposition 6, to show that the dual complexity space $(\mathcal{F}, d_{C^*})$ is hereditarily precompact.

It suffices to show that any subspace $(\mathcal{G}, d_{C^*})$ of (C^*, d_{C^*}), which has an upper bound, is precompact. No loss of generality results since all subspaces of $(\mathcal{F}, d_{C^*})$ are of this kind.

Let $(\mathcal{G}, d_{C^*})$ be a subspace of (C^*, d_{C^*}) which has an upper bound, say $m \in C^*$, and let ε be a strictly positive real number. Choose K to be a number such that

$\sum_{n>K} 2^{-n} m(n) < \frac{\varepsilon}{2}$. Note that for each $f \in \mathcal{G}$, $\sum_{n>K} 2^{-n} f(n) < \frac{\varepsilon}{2}$.

Since for each $n \leq K$ and each $f \in \mathcal{G}$, $f(n) \leq m(n)$, we obtain that, for each $n \leq K$ and each $f \in \mathcal{G}$, $f(n) \leq B$, where $B = max\{m(n) | \, n \leq K\}$.

Consider the set of functions $\mathcal{G}^K$ obtained from $\mathcal{G}$ by restricting each function of $\mathcal{G}$ to the domain $\{0, \ldots, K\}$.

For any $N \geq 1$, define a partition of the interval $[0, B]$ consisting of the intervals $B_0^N, \ldots, B_N^N$, where for $j \leq N$, $B_0^N = [0, \frac{B}{N+1}]$, and for each $j \geq 1$, $B_j^N = (j\frac{B}{N+1}, (j+1)\frac{B}{N+1}]$.

We identify functions which on every argument less than K have values which simultaneously belong to one of the intervals B_j^N. That is, we take the quotient of the set $\mathcal{G}^K$ by the equivalence relation $\approx$ defined by:

for all $f, g \in \mathcal{G}$, $f \approx g \Leftrightarrow [$ for each $i \leq K$ there is $j \leq N$ such that both $f(i), g(i) \in B_j^N]$.

The set $\mathcal{G}^K / \approx$ is obviously finite. Let its cardinality be n and choose n elements f_1,...,f_n of $\mathcal{G}$ such that $f_1|\{0, \ldots, K\}, \ldots, f_n|\{0, \ldots, K\}$ is a list of representatives, one for each class of the quotient $\mathcal{G}^K / \approx$.

Given $f \in \mathcal{G}$, let f_i be the representative such that $f_i|\{0, \ldots, K\} \approx f|\{0, \ldots, K\}$. Then

$$\begin{aligned} d_{C^*}(f_i, f) &= \sum_n 2^{-n} u(f_i(n), f(n)) \\ &= \sum_{n \leq K} 2^{-n} u(f_i(n), f(n)) + \sum_{n>K} 2^{-n} u(f_i(n), f(n)) \\ &< \sum_{n \leq K} 2^{-n} u(f_i(n), f(n)) + \frac{\varepsilon}{2}. \end{aligned}$$

Note that for any $n \leq K$ there exists a $j \leq N$ such that both $f_i(n), f(n) \in B_j^N$. So, for any $n \leq K$, $u(f_i(n), f(n)) \leq \frac{B}{N+1}$.

We choose N large enough such that $\frac{B}{N+1} < \frac{\varepsilon}{2(K+1)}$. Thus we obtain that $d_{C^*}(f_i, f) < \sum_{n \leq K} 2^{-n} u(f_i(n), f(n)) + \frac{\varepsilon}{2} < \sum_{n \leq K} \frac{\varepsilon}{2(K+1)} + \frac{\varepsilon}{2} \leq (K+1)\frac{\varepsilon}{2(K+1)} + \frac{\varepsilon}{2} = \varepsilon$. So the space $(\mathcal{F}, d_{C^*})$ is hereditarily precompact and thus totally bounded. □

Comments:

1) Note that the condition on the existence of an upper bound is necessary, as the following counterexample shows.

The dual complexity space (C^*, d_{C^*}) is not precompact and thus not totally bounded. By way of contradiction, we assume that, for every $\varepsilon > 0$ there exist $f_1, ..., f_n \in C^*$ such that for each $f \in C^*, d_{C^*}(f_i, f) < \varepsilon$ for some $i \in \{1, ...n\}$. Note that $d_{C^*}(f_i, f) \geq d_{C^*}(0, f) - d_{C^*}(0, f_i)$.

Given any $\varepsilon > 0$ and elements $f_1, \ldots, f_n$ of C^*, let c be the maximum of $d(0, f_1)$,..., $d(0, f_n)$ and pick $f \in C^*$ such that $d_{C^*}(0, f) \geq \varepsilon + c$. For instance, let f be the function with constant value k, large enough such that $\sum_{n=0}^{\infty} 2^{-n} k \geq \varepsilon + c$. We obtain a contradiction since, for each $i \in \{1, ...n\}$, $d_{C^*}(f_i, f) \geq d_{C^*}(0, f) - c \geq \varepsilon$.

2) The computational significance of the totally bounded spaces has been discussed in [14] by Smyth. In particular, spaces of programs which are only allowed to use limited resources are shown to correspond to totally bounded spaces.

As a complexity space with a lower bound corresponds to a program space where each program necessarily requires a minimum amount of resources, our approach seems to be the opposite of the one taken in [14].

This can be explained by the fact that the Denotational Semantics approach, presented in [14], and the Complexity Analysis approach, introduced here, take opposite viewpoints with respect to the ordering. In particular, a program which is undefined on all inputs will correspond to the minimum of the space in a Denotational Semantics context while, the same program in a Complexity Analysis context, corresponds to the maximum of the space.

However from a dual point of view, the apparent dissimilarity disappears and the above result as well as the original arguments of [14], illustrate that assumptions on bounds on resources and total boundedness of (topological) program spaces are tightly related.

For each $m \in C^*$ and for $\mathcal{F} \subseteq C^*$, we define $\mathcal{F}_m = \{f \in \mathcal{F} \mid m$ is an upper bound for $f\}$.

Theorem 9 *For each $m \in C^*$ and $\mathcal{F}$ closed in (C^*, d^s_{C*}), $(\mathcal{F}_m, d^s_{C^*})$ is a compact metric space.*

Proof: Let $m \in C^*$ and $\mathcal{F}$ closed in (C^*, d^s_{C*}). Then, by Theorem 8, $(\mathcal{F}_m, d_{C*})$ is a totally bounded quasi-metric space. On the other hand, one may verify that $\mathcal{F}_m$ is a closed subset of (C^*, d^s_{C*}).

Indeed, if $F \in \overline{\mathcal{F}_m}^{d^s_{C^*}}$ then there exists a sequence $(f_k)_{k\in\omega}$ in $\mathcal{F}_m$ such that $d^s_{C^*}(F, f_k) \to 0$. So, $F \in \mathcal{F}$. Moreover, for every $\varepsilon > 0$, there exists a k_ε such that $\sum_{n=0}^{\infty} 2^{-n}|F(n) - f_k(n)| < \varepsilon$ for all $k \geq k_\varepsilon$.

We may assume, proceeding by contradiction, that $F \notin \mathcal{F}_m$. Then, there is an index n_0 for which $F(n_0) > m(n_0)$. Let $\varepsilon = 2^{-n_0}(F(n_0) - m(n_0))$. Then there is an index k_{ε_0} such that $\sum_{n=0}^{\infty} 2^{-n}|F(n) - f_k(n)| < \varepsilon_0$ for all $k \geq k_{\varepsilon_0}$. However, this implies that $2^{-n_0}(F(n_0) - f_k(n_0)) < 2^{-n_0}(F(n_0) - m(n_0))$ and thus we obtain the contradiction $f_k(n_0) > m(n_0)$.

By Theorem 3, (C^*, d_{C^*}) is Smyth-complete and thus $(C^*, d^s_{C^*})$ is a complete metric space. We conclude that $(\mathcal{F}_m, d^s_{C^*})$ is a compact metric space. □

4 Contraction maps

We show that the applications of [12] can be obtained based on the dual complexity space.

We recall that, for applications, the complexity space is tipically restricted to functions which range over positive integers which are a power of a given integer b (we refer the reader to [12] for a motivation).

Let $a, b, c \in \omega$ be such that $a, b \geq 2$, let n range over the set $\{b^k \mid k \geq 0\}$ and let $h \in C$. A functional Φ, corresponding to a Divide & Conquer algorithm in the sense of [12], is tipically defined as follows:

$[\Phi(f)](n) = c$ when $n = 1$

$[\Phi(f)](n) = af(\frac{n}{b}) + h(n)$ when $n \in \{b^k \mid k \geq 1\}$.

We recall that this functional intuitively corresponds to a Divide & Conquer algorithm which recursively splits a given problem in a subproblems of size $\frac{n}{b}$ and which takes $h(n)$ time to recombine the separately solved problems into the solution of the original problem.

We consider, with slight abuse of notation, the inversion function Ψ on the reals defined by $\Psi:\mathcal{R}^+ \rightarrow (0, \infty]$, where $\Psi(x) = \frac{1}{x}$ for all $x \in \mathcal{R}^+$.

The corresponding functional Φ^* can then be defined on the dual C^* as follows:

$[\Phi^*(f)](n) = \Psi(c)$ when $n = 1$

$[\Phi^*(f)](n) = \Psi(a[\Psi(f)](\frac{n}{b}) + h(n))$ when $n \in \{b^k \mid k \geq 1\}$.

It is possible to give a "direct" definition of Φ^* on the dual C^* without using the inversion:

$[\Phi^*(f)](n) = \Psi(c)$ when $n = 1$

$[\Phi^*(f)](n) = \frac{\Psi(a)f(\frac{n}{b})\Psi(h(n))}{\Psi(a)f(\frac{n}{b})+\Psi(h(n))}$ when $n \in \{b^k \mid k \geq 1\}$.

We leave to the reader the straightforward verification that the definitions of Φ^* are equivalent.

It is clear that the computational interpretation of Φ in relation to Divide & Conquer algorithms is less obvious for the dual Φ^*. On the other hand, we recall that the dual complexity space (C^*, d_{C^*}) has the advantage of respecting the interpretation usually given to $\perp$ in a Denotational Semantics context and that its mathematical elegance enables a simplification of the proofs.

Finally, we remark that is is still possible to carry out complexity analysis in the dual context. Indeed, since Ψ is an isometry, we have that Φ is a contraction map on the complexity space (C, d_C) if and only if Φ^* is a contraction map on the dual complexity space (C, d_{C^*}).

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