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Complexity Spaces Revisited
(Extended Abstract)

M. Schellekens
Imperial College
Department of Computing
180 Queen's Gate, London SW7 2BZ, UK
E-mail: mpcs@doc.ic.ac.uk

Abstract

The complexity (quasi-pseudo-metric) spaces have been introduced as part of the development of a topological foundation for the complexity analysis of algorithms ([Sch95]). Applications of this theory to the complexity analysis of Divide & Conquer algorithms have been discussed in [Sch95]. Typically these applications involve fixed point arguments based on the Smyth completion.

The notion of S-completability plays an important role in the study of the complexity spaces since it implies a simplification of the Smyth completion to the bicompletion (e.g. [Sch95], [Smy92] and [Sün91]).

A characterization of S-completable quasi-uniform spaces has been given in [Sün91]. We present a related characterization of S-completable quasi-pseudo-metric spaces and provide a simplified proof of the fact that weightable quasi-pseudo-metric spaces are S-completable ([Kün93]).

We recall that the weightable quasi-pseudo-metric spaces and the totally bounded quasi-pseudo-metric spaces ([Smy91] and [Sün91]) form the two main examples of classes of S-completable quasi-pseudo-metric spaces encountered in the literature.

Complexity spaces are S-completable since they form a class of weightable spaces ([Sch95]). Complexity spaces with a complexity lower bound are shown to be totally bounded and this result is discussed in the light of Smyth's computational interpretation of totally boundedness ([Smy91]).

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1 Quasi-pseudo-metrics

General background

We use the following notation. $\mathcal{R}$ *denotes the set of real numbers and* $\mathcal{N}$ *denotes the set of natural numbers. We define* $\mathcal{R}^+ = (0, \infty)$ *and* $\mathcal{R}_0^+ = [0, \infty)$*, while* $\overline{\mathcal{R}} = \mathcal{R} \cup \{-\infty, \infty\}$, $\overline{\mathcal{R}^+} = \mathcal{R}^+ \cup \{\infty\}$ *and* $\overline{\mathcal{R}_0^+} = \mathcal{R}_0^+ \cup \{\infty\}$.

A function $d\colon X \times X \to \mathcal{R}_0^+$ *is a quasi-pseudo-metric iff*

1) $\forall x, y, z.\, d(x, y) + d(y, z) \geq d(x, z)$
2) $\forall x.\, d(x, x) = 0$.

The associated preorder $\leq_d$ *of a quasi-pseudo-metric* d *is defined by* $x \leq_d y$ *iff* $d(x, y) = 0$.

We recall the following useful fact (Lemma 5 of [Sch96]): if (X, d) *is a quasi-pseudo-metric space then* $\forall x, y, z \in X.\, (x' \leq_d x$ *and* $y' \geq_d y) \Rightarrow d(x', y') \leq d(x, y)$*. We will refer to this property as "the Monotonicity Lemma".*

The conjugate d^{-1} *of a quasi-pseudo-metric* d *is defined to be the function* $d^{-1}(x, y) = d(y, x)$*, which is again a quasi-pseudo-metric (e.g. [FL82]). The conjugate of a quasi-pseudo-metric space* (X, d) *is the quasi-pseudo-metric space* (X, d^{-1})*. The metric* d^* *induced by a quasi-pseudo-metric* d *is defined by* $d^*(x, y) = max\{d(x, y), d(y, x)\}$.

A preorder $(X, \leq)$ *is directed iff* $\forall x, y \in X\, \exists z \in X.\, z \geq x$ *and* $z \geq y$.

A quasi-pseudo-metric space is directed iff its associated preorder is directed. A quasi-pseudo-metric space (X, d) *has a maximum (minimum) iff the associated preordered space* $(X, \leq_d)$ *has a maximum (minimum).*

For any function $f\colon A \to B$ *and for any set* $X \subseteq A$, $f|X$ *indicates the restriction of* f *to the set* X*. A subspace of a quasi-pseudo-metric space* (X, d) *is a pair* $(Y, d|Y^2)$*, where* $Y \subseteq X$*. When no confusion can arise, we simplify the notation for a subspace* $(Y, d|Y^2)$ *of* (X, d) *by* (Y, d).

A function $f\colon (X, d) \to \mathcal{R}$ *is decreasing (increasing) iff* $\forall x, y \in X.\, x \leq_d y \Rightarrow f(x) \geq f(y) (f(x) \leq f(y))$.

Let $\Delta = \{(x, x) |\, x \in X\}$*. A partial order* $\leq$ *is discrete iff the order* $\leq$ *coincides with* Δ*. The topology* $\mathcal{T}_d$ *is* T_0 *iff* $\leq_d$ *is a partial order and is* T_1 *iff* $\leq_d$ *is a discrete order (e.g. [Mu66]).*

We implicitly assume in what follows that all quasi-pseudo-metric spaces we consider are T_0*; that is the spaces are assumed to have an associated order which is a partial order.*

A Cauchy net on a quasi-pseudo-metric space $(X,\, d)$ *is a net* $(x_\lambda)_{\lambda \in \Lambda}$ *such that* $\forall \epsilon > 0\, \exists \lambda_0\, \forall \nu \geq \mu \geq \lambda_0.\, d(x_\mu, x_\nu) < \epsilon$*. The terminology "forward (or left) Cauchy nets" is sometimes used to indicate Cauchy nets, as opposed to "backward (or right) Cauchy nets"; that is nets which are Cauchy with respect to* d^{-1}*. A biCauchy net on a quasi-pseudo-metric space* $(X,\, d)$ *is a net* $(x_\lambda)_{\lambda \in \Lambda}$ *such that* $\forall \epsilon > 0\, \exists \lambda_0\, \forall \mu, \nu \geq \lambda_0.\, d(x_\mu, x_\nu) < \epsilon$*. Note that a net* $(x_\lambda)_{\lambda \in \Lambda}$ *on* (X, d) *is biCauchy iff the net* $(x_\lambda)_{\lambda \in \Lambda}$ *is a Cauchy net on the metric space* (X, d^*)*. Cauchy and biCauchy sequences are* *defined in a similar way.*

A quasi-pseudo-metric space (X, d) *is bicomplete iff every biCauchy sequence on* (X, d) *converges with respect to* d^*. *A bicompletion of a quasi-pseudo-metric space* (X, d) *is a bicomplete quasi-pseudo-metric space* (X', d') *which has a* $\mathcal{T}(d^*)$*-dense subspace isometric to* (X, d). T_0 *quasi-pseudo-metric spaces have a unique (up to isometry)* T_0 *bicompletion ([Sal74]), indicated by "the bicompletion". For an introduction to the bicompletion, we refer the reader to [FL82].*

Examples

We give a few examples of quasi-pseudo-metric spaces which will be frequently referred to later on.

The function $d_1\colon \mathcal{R}^2 \to \mathcal{R}_0^+$, *defined by* $d_1(x, y) = y - x$ *when* $x < y$ *and* $d_1(x, y) = 0$ *otherwise, and its conjugate are quasi-pseudo-metrics. We refer to* d_1 *as the "left distance" and to its conjugate as the "right distance". These quasi-pseudo-metrics correspond to the nonsymmetric versions of the standard metric* m *on the reals, where* $\forall x, y \in \mathcal{R}.\, m(x, y) = |x - y|$.

The left distance induces the topology with a base consisting of the intervals $\{(-\infty, a) | \, a \in \mathcal{R}\}$ *while the right distance induces the topology with a base consisting of the intervals* $\{(a, \infty) | \, a \in \mathcal{R}\}$.

Note that the right distance has the usual order on the reals as associated order, that is $\forall x, y \in \mathcal{R}.\, x \leq_{d_1^{-1}} y \Leftrightarrow x \leq y$, *while for the left distance we have* $\forall x, y \in \mathcal{R}.\, x \leq_{d_1} y \Leftrightarrow x \geq y$.

The function $d_2\colon (\overline{\mathcal{R}} - \{0\})^2 \to \mathcal{R}_0^+$, *defined by* $d_2(x, y) = \frac{1}{y} - \frac{1}{x}$ *when* $y < x$ *and 0 otherwise, and its conjugate are quasi-pseudo-metrics.*

For any set $A \subset \mathcal{R}^+$ *and set* X, *a function* $f \in A^X$ *is bounded from below iff* $\exists c > 0\, \forall x \in X.\, f(x) \geq c$. *The set of functions of* A^X *which are bounded from below is denoted by* A_b^X.

This notation is generalized to quasi-pseudo-metric spaces $(\mathcal{R}^+, d)$ *as follows:*

for any set $A \subset \mathcal{R}^+$ *and set* X, *a function* $f \in A^X$ *is bounded from below iff* $\exists c > 0\, \forall x \in X.\, f(x) \geq_d c$. *The set of functions of* A^X *which are bounded from below is denoted by* A_b^X.

Let $\mathcal{C}$ *be the set of all functions from* $\overline{\mathcal{R}^+}^\omega$ *which are bounded from below, that is* $\mathcal{C} = \overline{\mathcal{R}^+}_b^\omega$.

We assume a basic familiarity with complexity measures for algorithms (e.g. [DW83]). An example of such a complexity measure is the "time" (or number of steps) an algorithm takes on each input to compute the corresponding output.

Each complexity measure associates with a given algorithm a complexity function. For the above example, the complexity function of an algorithm is the function which assigns to each natural number n, *which encodes an input for the algorithm, the time it takes to compute this input.*

The set $\mathcal{C}$ *includes all complexity functions which can occur in practice. Of course* *not every element of the space corresponds to a complexity function of a program ([Sch95]).*

The set $\mathcal{C}$ contains functions which take values in the reals, rather than the natural numbers, in order to include complexities obtained by average-case or asymptotic analysis (e.g. [Knu73]). The value ∞ for a complexity function of a program corresponds to an undefined output value for this program, which motivates the inclusion of infinity as a possible function value.

Definition 1 *The complexity distance d on a set $X \subseteq \mathcal{C}$ is defined by:*

$$\forall f, g \in X.\, d(f, g) = \sum_{n \geq 0} \{ (\frac{1}{g(n)} - \frac{1}{f(n)}) \frac{1}{2^n} |\, f(n) > g(n) \}.$$

Complexity spaces are directed subspaces of $(\mathcal{C}, d)$. Since $(\mathcal{C}, d)$ will be our main object of study we sometimes will refer to this space as "the" complexity space.

We remark that the associated order of the complexity space $(\mathcal{C}, d)$ is the pointwise order on functions, which has $\top$ as maximum.

The requirement on the directedness of a complexity space is motivated in [Sch96a]. Basically this corresponds to the fact that for any two programs which compute a given partial recursive function and for any given complexity measure, there always exists a "worse" program computing the same function, which has larger complexity (in the pointwise order).

Function spaces

For any set A, we define $A^\omega = \{f |\, f\colon \mathcal{N} \to A\}$. We will consider function spaces which are obtained from a given quasi-pseudo-metric space (X, d), of the kind $(\mathcal{F}, d^\omega)$, where $\mathcal{F} \subseteq X^\omega$ and where the distance d^ω is defined by $\forall f, g \in \mathcal{F}.\, d^\omega(f, g) = \sum_n d(f(n), g(n)) \frac{1}{2^n}$, on condition that the sum converges on the elements of $\mathcal{F}$. In what follows, whenever a function space is considered, we implicitly assume that this condition is satisfied.

For instance, in case d is bounded, that is $\exists K \in \mathcal{R}_0^+\, \forall x, y \in X.\, d(x, y) \leq K$, the distance d^ω is defined on the entire set X^ω.

It is easy to verify that a function space $(\mathcal{F}, d^\omega)$ has the pointwise order generated from $\leq_d$ as associated order, that is $\forall f, g \in \mathcal{F}.\, f \leq_{d^\omega} g \Leftrightarrow \forall n. f(n) \leq_d g(n)$.

A subspace $(\mathcal{G}, d^\omega)$ of a given function space $(\mathcal{F}, d^\omega)$ has a lower bound $m \in \mathcal{F}$ iff $\forall n\, \forall f \in \mathcal{G}.\, m(n) \leq_d f(n)$.

In case the associated order of a quasi-pseudo-metric space is a linear order we refer to the space as a linear quasi-pseudo-metric space.

Weighted spaces

We recall that the generalized metric spaces known as the "weightable quasi-pseudo-metric spaces" have been introduced by Matthews in [Mat94] and [Mat92a] as part of *the study of the denotational semantics of dataflow networks (cf. also [Kah74] and [Wad81]).*

Definition 2 *A quasi-pseudo-metric space (X, d) is weightable iff there exists a function $w: X \rightarrow \mathcal{R}_0^+$ such that $\forall x, y \in X.\, d(x, y) + w(x) = d(y, x) + w(y)$. The function w is called a weighting function, $w(x)$ is the weight of x and the quasi-pseudo-metric d is weightable by the function w. A weighted space is a triple (X, d, w) where (X, d) is a quasi-pseudo-metric space weightable by the function w. If (X, d) is a quasi-pseudo-metric space then (X, d) is upper weightable iff there exists a weighting function w for (X, d) such that $\forall x, y \in X.\, d(x, y) \leq w(y)$. We refer to such a function w as an upper weighting function. A weighted space (X, d, w) is upper weighted iff w is an upper weighting function.*

Examples: The quasi-pseudo-metric space $(\mathcal{R}_0^+, d_1)$ is weightable by the identity function, $w_1(x) = x$ and the space $(\overline{\mathcal{R}^+}, d_2)$ is weightable by the function $w_2(x) = \frac{1}{x}$. The Baire quasi-pseudo-metric spaces form an example of a class of weightable spaces. Complexity spaces (X, d) are weightable by the weighting w defined by: $\forall f \in X.\, w(f) = \sum_n \frac{1}{f(n)} \frac{1}{2^n}$ ([Sch95]).

Let $S^{\leq\omega}$ be the set of countably infinite and finite sequences of elements from a given set S and let $\emptyset$ be the empty sequence. Given a sequence $s \in S^{\leq\omega}$, say of length $L \geq 1$, then for any natural number n such that $1 \leq n \leq L$, $s(n)$ denotes the n-th element of the sequence. Define the function $p: S^{\leq\omega} \times S^{\leq\omega} \rightarrow \mathcal{R}_0^+$ as follows,

> *$\forall x, y \in S^{\leq\omega}.\, p(x, y) = 2^{-\alpha}$, where $\alpha = max\{n \mid x(n) = y(n)\}$ when the sequences x and y have a common non empty initial segment and $\alpha = 0$ otherwise.*

The function p is a "partial metric", that is a generalized metric which does not necessarily satisfies the axiom of reflexivity "$\forall x.\, p(x, x) = 0$". The function is also referred to as the "Baire partial metric" or the "Kahn partial metric" (e.g. [Kah74],[Mat92a] and [Wad81]). We recall that any partial metric p can be replaced by a weightable quasi-pseudo-metric d which induces the same topology, where d is defined by: $d(x, y) = p(x, y) - p(x, x)$ and where a weighting function for d is given by the function $w(x) = p(x, x)$ (cf. [Mat94]).

The above examples, except for the Baire quasi-pseudo-metric spaces, provide examples of upper weighted spaces.

Not every weightable quasi-pseudo-metric space has a weightable conjugate. For instance the quasi-pseudo-metric space $(\mathcal{R}_0^+, d_1^{-1})$ is not weightable ([Sch96]).

2 S-completability

In order to clarify the notion of S-completability, originally introduced in [Sün91], we need to make a few comments on the theory of (topological) quasi-uniform spaces, of which no prior knowledge is required for this paper.

The quasi-uniform spaces generalize the quasi-pseudo-metric spaces and have been extensively studied in [FL82]. The topological quasi-uniform spaces have been introduced in [Sün91] as a category extending the quasi-uniform spaces ([FL82]) which provides a suitable basis on which to develop the Smyth completion. We recall that the Smyth completion provides a topological foundation for Denotational Semantics, and in particular for the well know cpo-completion (e.g. [Smy89], [Smy92] and [Sün91]).

The S-completable (topological) quasi-uniform spaces have been defined in [Sün91] as the (topological) quasi-uniform spaces of which the Smyth completion is again a quasi-uniform space; a condition which in general is violated as indicated in [Sün91].

Apart from forming a class with nice closure properties with respect to the Smyth completion, the S-completable spaces can be interpreted to form a class of nonsymmetric spaces which still posses an "inherent symmetry"; that is to form a class of "weakly symmetric" spaces.

This interpretation is based on a characterization of S-completable (topological) quasi-uniform spaces in terms of Cauchy nets (Theorem 5 of [Sün95]), which we discuss below.

We adopt this characterization in what follows as an alternative definition of the S-completable spaces, as this approach does not require any reference to the more abstract context of the theory of topological quasi-uniform spaces.

The definition given below is based on an adaptation of this characterization to the specific case of the quasi-pseudo-metric spaces, which suffices for our purposes.

Definition 3 *A quasi-pseudo-metric space (X, d) is S-completable iff every Cauchy net on (X, d) is biCauchy.*

The S-completable quasi-pseudo-metric spaces are weakly symmetric in the sense that any net which is a Cauchy net with respect to the quasi-pseudo-metric d is also a Cauchy net with respect to the metric d^*.

The weakly symmetric nature of the spaces is illustrated by the fact that some properties of metric spaces generalize, under suitable hypotheses, to the context of S-completable spaces (e.g. Proposition 8 of Künzi). This intuition also lies at the basis of the fact that for S-completable spaces, the Smyth completion simplifies to the bicompletion ([Sün91]).

We show that the S-completability condition can be simplified for the case of quasi-pseudo-metric spaces to a requirement on sequences rather than on nets.

Proposition 4 *A quasi-pseudo-metric space is S-completable iff every Cauchy sequence on the space is biCauchy.*

Proof: *Let (X, d) be a quasi-pseudo-metric space. It suffices to show that when every Cauchy sequence on (X, d) is biCauchy, the space (X, d) is S-completable.*

We assume by way of contradiction that every Cauchy sequence on the space is biCauchy, but that the space is not S-completable. Then there exists a Cauchy net $(x_\lambda)_{\lambda\in\Lambda}$ which is not biCauchy, that is

(1) $\forall\epsilon > 0\,\exists\lambda_0\,\forall\nu \geq \mu \geq \lambda_0.\, d(x_\mu, x_\nu) < \epsilon$.
(2) $\exists\epsilon_0 > 0\,\forall\lambda\,\exists\mu, \nu \geq \lambda.\, d(x_\mu, x_\nu) \geq \epsilon_0$.

Consider $(\epsilon_i)_{i\geq 1}$, a strictly decreasing sequence such that $\epsilon_1 < \epsilon_0$ and with limit 0. We define a sequence $(x_{\mu_j}, x_{\nu_j})_{j\geq 1}$ by induction as follows:

a) For ϵ_1, obtain λ_1 via (1), and obtain x_{μ_1}, x_{ν_1} via (2) for λ_1.
b) Assume that $(x_{\mu_j}, x_{\nu_j})_{1\leq j\leq k}$ has been defined.

For ϵ_{k+1}, obtain λ'_{k+1} via (1). Consider an index λ_{k+1} such that $\lambda_{k+1} \geq \lambda'_{k+1}$, μ_k, ν_k and obtain $x_{\mu_{k+1}}$ and $x_{\nu_{k+1}}$ via (2) for λ_{k+1}.

The reader may find it helpful to refer to Figure 1 below.

An arrow between points, say x_α and x_β, indicates that we measure the distance from x_α to x_β. Note that we only indicate arrows between points x_α and x_β, when $\alpha \leq \beta$. An index ϵ_i attached to an arrow indicates that the distance from x to y is less than ϵ_i.

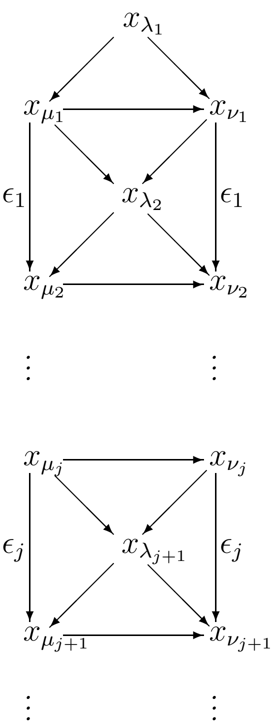

Figure 1

The sequence $(x_{\mu_j}, x_{\nu_j})_{j\geq 1}$ *satisfies the following property:*

(3) $\forall j \geq 1.\, d(x_{\mu_j}, x_{\nu_j}) \geq \epsilon_0$.

(Cf. the horizontal arrows of Figure 1.)

Note that in replacing λ'_{k+1} *by a larger index* λ_{k+1}*, we have that (1) is still satisfied where* λ_{k+1} *dominates both* μ_k *and* ν_k*; a property which is not necessarily guaranteed for* λ'_{k+1}*. This last fact together with the fact that* $\forall j \geq 1.\, \mu_j, \nu_j \geq \lambda_j$ *is represented by the diagonal arrows of Figure 1.*

So we obtain the following inequalities: $\forall j \geq 1.\, \mu_{j+1}, \nu_{j+1} \geq \mu_j, \nu_j \geq \lambda_j$*. Thus in particular, by (1), we have that the sequence* $(x_{\mu_j}, x_{\nu_j})_{j\geq 1}$ *satisfies the following properties:*

(4) $\forall j \geq 1.\, d(x_{\mu_j}, x_{\mu_{j+1}}) < \epsilon_j$ *and* $d(x_{\nu_j}, x_{\nu_{j+1}}) < \epsilon_j$
(5) $\forall j \geq 1.\, d(x_{\mu_j}, x_{\nu_{j+1}}) < \epsilon_j$ *and* $d(x_{\nu_j}, x_{\mu_{j+1}}) < \epsilon_j$.

(The vertical arrows of Figure 1 represent the inequalities displayed under (4).)

We remark that the sequence $(x_{\mu_j})_{j\in\mathcal{N}}$ *is a Cauchy sequence. We show that the sequence is not biCauchy. Note that* $\forall\, j \geq 1.\, d(x_{\mu_{j+1}}, x_{\mu_j}) + d(x_{\mu_j}, x_{\nu_{j+1}}) \geq d(x_{\mu_{j+1}}, x_{\nu_{j+1}})$ *and thus* $d(x_{\mu_{j+1}}, x_{\mu_j}) \geq d(x_{\mu_{j+1}}, x_{\nu_{j+1}}) - d(x_{\mu_j}, x_{\nu_{j+1}})$.

By (3) and (5) we have that $d(x_{\mu_{j+1}}, x_{\mu_j}) \geq \epsilon_0 - \epsilon_j \geq \epsilon_0 - \epsilon_1$*. So we obtain:*

(6) $\forall j \geq 1.\, d(x_{\mu_{j+1}}, x_{\mu_j}) \geq \epsilon$*, where* $\epsilon = \epsilon_0 - \epsilon_1$.

So the sequence $(x_{\mu_j})_{j\in\mathcal{N}}$ *is not biCauchy. Thus we obtain a contradiction, which implies that the space* (X, d) *is S-completable.*

□

3 Weightable spaces

Two main examples of classes of S-completable quasi-pseudo-metric spaces have been discussed in the literature: the weightable spaces (e.g. [KV93]) and the totally bounded spaces (e.g. [Sün91]). In [Kün93] it is shown that every totally bounded quasi-pseudo-metric space (X, d) *can be replaced by an equivalent*[1] *weightable space* (X, d')*. Hence the weightable quasi-pseudo-metric spaces include all cases of S-completable quasi-pseudo-metric spaces thus far encountered in the literature.*

The S-completability of weightable quasi-pseudo-metric spaces has been shown by Künzi in [Kün93] (Proposition 15).

[1] "Equivalent" in the sense that the quasi-pseudo-metrics involved generate the same quasi-uniformity. For an introduction to the theory of quasi-uniform spaces, we refer the reader to [FL82].

An informal motivation for the fact that weightable spaces are S-completable can be given by the following observation. For any weightable space (X, d)*, say with a weighting function* w*, the weighting equality,* $\forall x, y \in X. d(x,y)+w(x) = d(y,x)+w(y)$*, can be interpreted to reflect a control of the nonsymmetry, in the sense that the lost symmetry equality* $\forall x, y \in X. d(x,y) = d(y,x)$ *can be "replaced" by a new equality via the addition of weighting factors. "Hence" weightable spaces form a class of weakly symmetric spaces and thus are S-completable.*

We present an alternative simplified proof of Künzi's result, via an argument based on Proposition 4.

Proposition 5 (*Künzi, [Kün93]*) *Weightable quasi-pseudo-metric spaces are S-completable.*

Proof: *Let* (X, d) *be a weightable quasi-pseudo-metric space, say with a weighting function* w*. We assume, by way of contradiction, that there exists a Cauchy sequence* $(x_n)_n$ *which is not biCauchy. Thus there exists an* $\epsilon_0 > 0$ *such that* $\forall n\, \exists k, l \geq n. d(x_k, x_l) \geq \epsilon_0$.

Since the sequence is a Cauchy sequence we can obtain an index p *such that*

$$(1)\, \forall k \geq l \geq p. d(x_l, x_k) < \frac{\epsilon_0}{4}.$$

Hence we also obtain that

$$(2)\, \forall n \geq p\, \exists k \geq l \geq n. d(x_k, x_l) \geq \epsilon_0.$$

Via (2), we define a subsequence $(y_n)_n$ *of* $(x_n)_n$ *inductively as follows:*

for $n = p$ *we obtain indices* k_1, l_1 *such that* $k_1 \geq l_1 \geq p$ *and* $d(x_{k_1}, x_{l_1}) \geq \epsilon_0$.

Under the assumption that k_i, l_i *have been defined for* $i\colon 1 \ldots N$*, where* $N \geq 1$*, we define indices* k_{i+1}, l_{i+1} *such that* $k_{i+1} \geq l_{i+1} \geq k_i$ *and* $d(x_{k_{i+1}}, x_{l_{i+1}}) \geq \epsilon_0$.

The sequence $(y_n)_n$ *is defined by:* $\forall n \geq 0. y_{2n+1} = x_{l_{n+1}}$ *and* $y_{2n+2} = x_{k_{n+1}}$*. Then* $\forall n \geq 0. w(y_{2n+1}) - w(y_{2n+3}) = w(y_{2n+1}) - w(y_{2n+2}) + w(y_{2n+2}) - w(y_{2n+3}) = d(y_{2n+2}, y_{2n+1}) - d(y_{2n+1}, y_{2n+2}) + d(y_{2n+3}, y_{2n+2}) - d(y_{2n+2}, y_{2n+3}) \geq d(y_{2n+2}, y_{2n+1}) - d(y_{2n+1}, y_{2n+2}) - d(y_{2n+2}, y_{2n+3}) \geq \epsilon_0 - \frac{\epsilon_0}{4} - \frac{\epsilon_0}{4} = \frac{\epsilon_0}{2}$.

So the sequence $(z_n)_n$ *defined by* $\forall n \geq 0. z_n = y_{2n+1}$ *satisfies the following:*

$$\forall n \geq 0. w(z_j) \geq w(z_{j+1}) + \frac{\epsilon_0}{2}$$

and thus $\forall n \geq 1. w(z_0) \geq w(z_n) + n\frac{\epsilon_0}{2}$*. So* $w(z_0) = \infty$*, which is a contradiction. Hence every Cauchy sequence is biCauchy and thus, by Proposition 4,* (X, d) *is S-completable.*

□

Corollary 6 *Complexity spaces are S-completable.*

Proof: *This follows by Proposition 5 and by the fact that complexity spaces are weightable ([Sch95]).*

□

4 Totally boundedness

We recall the definitions of precompact and totally bounded quasi-pseudo-metric spaces ([FL82]). As mentioned above the totally bounded spaces form an example of S-completable spaces ([Sün95]) and their relevance to Computer Science and in particular to Complexity Theory has been discussed in [Smy91].

Definition 7 *Given a quasi-pseudo-metric space* (X, d),

1) (X, d) *is precompact iff* $\forall \epsilon > 0\, \exists x_1, \ldots, x_n\, \forall x\, \exists i.\, d(x_i, x) < \epsilon$.
2) (X, d) *is totally bounded iff* $\forall \epsilon > 0\, \exists x_1, \ldots, x_n\, \forall x\, \exists i.\, d^*(x_i, x) < \epsilon$.

Note that totally boundedness implies precompactness and that for metric spaces, that is for the symmetric case, the notions of totally boundedness and precompactness coincide.

This is not necessarily true for the nonsymmetric case. A counterexample is for instance given by the space $(\mathcal{R}_0^+, d_1^{-1})$. *The space is precompact since* $\forall x \geq 0.\, d(0, x) = 0$ *but is not totally bounded since the associated metric* $(d_1^{-1})^*$ *is the standard metric* m *on the positive reals.*

The above result for the symmetric case has been extended by Künzi ([Kün93], Proposition 12) to a "weakly symmetric" context, by the following result on S-completable spaces, which we formulate in terms of quasi-pseudo-metric spaces.

Proposition 8 (*Künzi*) *Every hereditarily precompact S-completable quasi-pseudo-metric space is totally bounded.*

We show that complexity spaces are totally bounded under the assumption that they have a lower bound.

This assumption can be motivated as follows. In general, for a given problem, a minimum amount of complexity will be required in order for any program to carry out the prescribed task. That is, for a class of programs calculating a given partial recursive function, there will typically exist a lower bound on the complexity of the programs. Hence many complexity arguments involve the determination of a complexity lower bound (e.g. [Knu73]), which makes the study of complexity spaces with a lower bound a worthwhile goal.

We will consider spaces $(\mathcal{F}, d^\omega, w^\omega)$ *which are subspaces of a weighted function space* $(\overline{\mathcal{R}^+}_b^\omega, d^\omega, w^\omega)$ *generated from a given weighted quasi-pseudo-metric space* (X, d, w) *(cf. Section 1), where* $\forall f \in \overline{\mathcal{R}^+}_b^\omega.\, w^\omega(f) = \sum_n w(f(n)) \frac{1}{2^n}$.

Since in practice complexity values are assumed to be comparable, we will assume in the following that the function space $(\overline{\mathcal{R}^+}_b^\omega, d^\omega, w^\omega)$ *is generated by a linear quasi-pseudo-metric space* $(\overline{\mathcal{R}^+}, d, w)$.

The complexity space $(\mathcal{C}, d, w)$ *as discussed in section 1 is an example of a function space generated as above, by the weighted linear quasi-pseudo-metric space* $(\overline{\mathcal{R}^+}, d_2, w_2)$.

We show that function spaces, as defined above, are well defined (cf. also [Sch96]).

If $g \in \overline{\mathcal{R}^+}^{\omega}_b$ then there exists a constant $c > 0$ such that $\forall n.\, g(n) \geq_d c$.

We remark that since the function space $(\overline{\mathcal{R}^+}^{\omega}_b, d^\omega, w^\omega)$ is generated from a linear quasi-pseudo-metric space and has as associated order the pointwise order generated from the order $\leq_d$, this function space is directed. Since any weighted directed space is upper weighted ([Sch96]), we obtain that $\forall f \in \overline{\mathcal{R}^+}^{\omega}_b.\, d^\omega(f,g) = \sum_n d(f(n), g(n))\frac{1}{2^n} \leq \sum_n w(g(n))\frac{1}{2^n} \leq \sum_n w(c)\frac{1}{2^n}$ by the fact that upper weighting functions are decreasing. This fact is shown in [Sch96b].

For the sake of completeness we show this fact for the special case of linear weighted spaces (X, d, w). Assume that $x, y \in X$ are elements such that $x \geq_d y$ then $d(x,y) = w(y) - w(x)$ and thus $w(y) - w(x) \geq 0$, which implies $w(x) \leq w(y)$. In case that $x \leq_d y$ we obtain that $d(y,x) = w(x) - w(y) \geq 0$ and thus $w(x) \geq w(y)$. Hence the weighting function w is decreasing.

So we have shown that the distance d^ω converges. A similar argument shows that the weighting function w^ω converges.

Theorem 9 *Let $(\overline{\mathcal{R}^+}, d, w)$ be a weighted linear quasi-pseudo-metric space. If $(\mathcal{F}, d^\omega, w^\omega)$ is a subspace of $(\overline{\mathcal{R}^+}^{\omega}_b, d^\omega, w^\omega)$ which has a lower bound then $(\mathcal{F}, d^\omega, w^\omega)$ is totally bounded.*

Proof: *Let $(\mathcal{F}, d^\omega, w^\omega)$ be a subspace of $(\overline{\mathcal{R}^+}^{\omega}_b, d^\omega, w^\omega)$ which has a lower bound, say $m \in \overline{\mathcal{R}^+}^{\omega}_b$. Since the space $(\mathcal{F}, d^\omega, w^\omega)$ is weighted, it is S-completable and thus it suffices to show by Proposition 8 (Künzi) that the space is hereditarily precompact.*

It suffices to show that any subspace $(\mathcal{G}, d^\omega, w^\omega)$ of $(\overline{\mathcal{R}^+}^{\omega}_b, d^\omega, w^\omega)$ which has a lower bound, is precompact. No loss of generality results since all subspaces of $(\mathcal{F}, d^\omega, w^\omega)$ are of this kind.

Let $(\mathcal{G}, d^\omega, w^\omega)$ be a subspace of $(\overline{\mathcal{R}^+}^{\omega}_b, d^\omega, w^\omega)$ which has a lower bound, say $m \in \overline{\mathcal{R}^+}^{\omega}_b$, and let ϵ be a strictly positive real number.

Choose K to be a number such that $\sum_{n>K} w(m(n))\frac{1}{2^n} < \frac{\epsilon}{2}$.

Note that $\forall f \in \mathcal{G}.\, \sum_{n>K} w(f(n))\frac{1}{2^n} < \frac{\epsilon}{2}$. This follows from the fact that w is decreasing and thus $\forall n.\, w(f(n)) \leq w(m(n))$.

Since $\forall n \leq K\, \forall f \in \mathcal{G}.\, w(f(n)) \leq w(m(n))$ we obtain that $\forall n \leq K\, \forall f \in \mathcal{G}.\, w(f(n)) \leq B$ where $B = max\{w(m(n))\,|\, n \leq K\}$.

Consider the set of functions $\mathcal{G}^K$ obtained from $\mathcal{G}$ by restricting each function of $\mathcal{G}$ to the domain $\{0, \ldots, K\}$.

For any $N \geq 1$, define a partition of the interval $[0, B]$ consisting of the intervals $B^N_0, \ldots, B^N_N$, where for $j \leq N$, $B^N_0 = [0, \frac{B}{N+1}]$ and $\forall j \geq 1.\, B^N_j = (j\frac{B}{N+1}, (j+1)\frac{B}{N+1}]$.

We identify functions which on every argument less than K have values for which the weights simultaneously belong to one of the intervals B^N_j.

That is, we take the quotient of the set $\mathcal{G}^K$ by the equivalence relation $\approx$ defined by: $\forall f, g \in \mathcal{G}.\, f \approx g \Leftrightarrow [\forall i \leq K\, \exists j \leq N.\, w(f(i)), w(g(i)) \in B^N_j]$.

The set $\mathcal{G}^K/\approx$ is obviously finite. Let its cardinality be n and choose n elements $f_1, \ldots, f_n$ of Y such that $f_1|\{0, \ldots, K\}, \ldots, f_n|\{0, \ldots, K\}$ is a list of representatives, *one for each class of the quotient $\mathcal{G}^K/\approx$.*

Given $f \in Y$, let f_i be the representative such that $f_i|\{0,\ldots,K\} \approx f|\{0,\ldots,K\}$, then

$$\begin{aligned} d^\omega(f_i, f) &= \sum_n d(f_i(n), f(n))\frac{1}{2^n} \\ &= \sum_{n\le K} d(f_i(n), f(n))\frac{1}{2^n} + \sum_{n>K} d(f_i(n), f(n))\frac{1}{2^n} \\ &< \sum_{n\le K} d(f_i(n), f(n))\frac{1}{2^n} + \frac{\epsilon}{2}. \end{aligned}$$

Note that for any $n \le K$ we have that $\exists j \le N.\, w(f_i(n)), w(f(n)) \in B_j^N$.

So $\forall n \le K.\, d(f_i(n), f(n)) = d_1(w(f_i(n)), w(f(n)) \le |w(f_i(n)) - w(f(n))| \le \frac{B}{N+1}$, where the first equality follows by the fact that $(\overline{\mathcal{R}^+}, d)$ is a linear weighted quasi-pseudo-metric space.

We choose N large enough such that $\frac{B}{N+1} < \frac{\epsilon}{2(K+1)}$. Thus we obtain that $d^\omega(f_i, f) < \sum_{n\le K} d(f_i(n), f(n))\frac{1}{2^n} + \frac{\epsilon}{2} < \sum_{n\le K} \frac{\epsilon}{2(K+1)} + \frac{\epsilon}{2} \le (K+1)\frac{\epsilon}{2(K+1)} + \frac{\epsilon}{2} = \epsilon$. So the space $(\mathcal{F}, d^\omega, w^\omega)$ is hereditarily precompact and thus totally bounded.

□

Comments:

1) Note that the condition on the existence of a lower bound is needed as the following counterexample shows.

The complexity space $(\mathcal{C}, d)$ is not precompact and thus not totally bounded.

By way of contradiction, assume that $\forall \epsilon > 0\, \exists f_1, ..., f_n \in \mathcal{C}\, \forall f \in \mathcal{C}\, \exists i \in \{1,\ldots,n\}.\ d(f_i, f) < \epsilon$. Note that $d^\omega(f_i, f) \ge w(f) - w(f_i)$ by weightedness of $(\mathcal{C}, d)$. Given any $\epsilon > 0$ and elements $f_1,\ldots,f_n$ of $\mathcal{C}$, let c be the maximum of the weights of $f_1,\ldots,f_n$ and pick $f \in \mathcal{C}$ such that $w(f) \ge \epsilon + c$. For instance let f be the function with constant value d, small enough such that $\sum_n \frac{1}{d2^n} \ge \epsilon + c$. We obtain a contradiction since $\forall i : 1\ldots n.\, d(f_i, f) \ge w(f) - c \ge \epsilon$.

2) The computational significance of the totally bounded spaces has been discussed in [Smy91] by Smyth. In particular, spaces of programs which are only allowed to use limited resources are shown to correspond to totally bounded spaces.

As a complexity space with a lower bound corresponds to a program space where each program necessarily requires a minimum amount of resources, our approach is the opposite of the one taken in [Smy91].

This can be explained by the fact that the Denotational Semantics approach, presented in [Smy91], and the Complexity Analysis approach, introduced here, take opposite viewpoints with respect to the ordering. In particular, a program which is undefined on all inputs will correspond to the minimum of the space in a Denotational Semantics context while the same program in a Complexity Analysis context corresponds to the maximum of the space.

Whatever point of view one takes (Denotational Semantic or Complexity Theoretic), the above result and the original arguments of [Smy91], illustrate that assumptions on bounds on resources and totally boundedness of (topological) program spaces are tightly related.

Corollary 10 *Complexity spaces with a lower bound have a compact Smyth-completion.*

Proof: *We recall the following proposition on the bicompletion (Proposition 3.36 of [FL82]), stated in terms of quasi-pseudo-metric spaces.*

A T_0 quasi-pseudo-metric space is totally bounded iff its bicompletion is compact.

The result follows immediately by the fact that the complexity spaces are S-completable (Corollary 6) and by the fact that the Smyth completion of an S-completable space coincides with its bicompletion (Corollary 8 of [Sün91]).

□

Note that by Proposition 3.36 and comment 1) above, the condition on the existence of a lower bound is necessary in order to guarantee compactness.

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