On upper weightable spaces

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Abstract

The weightable quasi-pseudo-metric spaces have been introduced by Matthews as part of the study of the denotational semantics of dataflow networks (e.g. [Mat92] and [Mat92a]). The study of these spaces has been continued in the context of Nonsymmetric Topology by Kunzi and Vajner ([KV93] and [Kün93]). We introduce and motivate the class of upper weightable quasi-pseudo-metric spaces. The relationship with the development of a topological foundation for the complexity analysis of programs ([Sch95]) is discussed, which leads to the study of the weightable optimal (quasi-pseudo-metric) join semilattices.

1 Introduction

Weightable quasi-pseudo-metric spaces have been introduced by Matthews in the context of the study of the Denotational Semantics of dataflow networks (e.g. [Kah74], [Mat92],[Mat92a] and [Wad81]).

We recall that the weightable quasi-pseudo-metrics originally have been introduced as "partial metrics", which are a new kind of generalized metrics for which the "reflexivity-axiom", $\forall x. d(x,x) = 0$, is not required to hold. Since the partial metric spaces have been shown to be equivalent to the weightable quasi-pseudo-metric spaces ([Mat92]), their topological study can be carried out in the context of Nonsymmetric Topology.

The study of the weightable quasi-pseudo-metric spaces has been the subject of [KV93] and also of the survey paper "Nonsymmetric Topology" ([Kün93]), where several open characterization problems on weightable spaces have been stated[1]: "Characterize those quasi-uniformities having a countable base which are induced by a weighted quasi-pseudo-metric" (Problem 7), "Which topological spaces admit weightable quasi-pseudo-metrics?" (Problem 8) and "Develop a concept of a weighted quasi-uniformity" (Problem 10).

For the open problems stated above, some partial results in connection to Problem 7 and Problem 8 are known ([KV93] and [Kün93]). The results are limited to

[1] In [KV93] the problems are actually stated in terms of "quasi-metrics", which correspond to the "quasi-pseudo-metrics" as originally defined in [FL82].

restricted classes as for instance the class of the Alexandroff topologies in connection to a partial solution of problem 8 ([KV93]). In [Kün93] the remark is made in connection to problem 7 that any totally bounded quasi-uniform space with a countable base can be induced by a weighted quasi-pseudo-metric.

The paper introduces and motivates a sub class of the weightable spaces, the upper weightable spaces. A characterization of the upper weightable spaces has been obtained in [Sch95a], which is briefly discussed in the paper (no results of the present paper are based on this characterization). The relevance of these spaces with respect to the development of a topological foundation for the complexity analysis of programs ([Sch95]) is discussed, which leads to the introduction of the class of the weightable optimal (quasi-pseudo-metric) join semilattices.

These upper weightable quasi-pseudo-metric spaces posses by definition an associated partial order which is a join semilattice for which the join operation is "optimal" with respect to the distance function (in a sense explained below).

The paper presents two characterization results for this class of spaces.

2 Quasi-pseudo-metric spaces

Quasi-pseudo-metrics are generalized metrics which do not necessarily satisfy the axiom of symmetry. Their study belongs to the field of Nonsymmetric Topology (e.g. [FK93],[FL82],[KV93] and [Kün93]) and these generalized metrics have for instance been applied in the context of Denotational Semantics (e.g. [Kah74],[Mat92],[Mat92a], [Smy87] and [Wad81]) as well as in Complexity Theory (e.g. [Sch95] and [Smy91]).

We use the following notation. $\mathcal{R}$ denotes the set of real numbers and $\mathcal{N}$ denotes the set of natural numbers. We define $\mathcal{R}^+ = (0, \infty)$ and $\mathcal{R}_0^+ = [0, \infty)$, while $\overline{\mathcal{R}^+} = \mathcal{R}^+ \cup \{\infty\}$ and $\overline{\mathcal{R}_0^+} = \mathcal{R}_0^+ \cup \{\infty\}$.

A function $d: X \times X \rightarrow \mathcal{R}_0^+$ is a *quasi-pseudo-metric* iff

1) $\forall x, y, z.\, d(x, y) + d(y, z) \geq d(x, z)$
2) $\forall x.\, d(x, x) = 0$.

A quasi-pseudo-metric d is a *quasi-metric* iff

3) $\forall x, y.\, d(x, y) = 0 \Rightarrow x = y$.

We give a few examples of quasi-pseudo-metric spaces which will be frequently referred to later on.

Examples: The function $d_1: \mathcal{R}^2 \rightarrow \mathcal{R}_0^+$, defined by $d_1(x, y) = y - x$ when $x < y$ and $d_1(x, y) = 0$ otherwise, and its conjugate are quasi-pseudo-metrics. We refer to d_1 as the "left distance" and to its conjugate as the "right distance" (e.g. [Kün93]). These quasi-pseudo-metrics correspond to the nonsymmetric versions of the standard metric m on the reals, where $\forall x, y \in \mathcal{R}.\, m(x, y) = |x - y|$.

The left distance induces the topology with a base consisting of the intervals $\{(-\infty, a) | \, a \in \mathcal{R}\}$ while the right distance induces the topology with a base consisting of the intervals $\{(a, \infty) | \, a \in \mathcal{R}\}$.

Note that the right distance has the usual order on the reals as associated order, that is $\forall x, y \in \mathcal{R}. \, x \leq_{d_1^{-1}} y \Leftrightarrow x \leq y$, while for the left distance we have $\forall x, y \in \mathcal{R}. \, x \leq_{d_1} y \Leftrightarrow x \geq y$.

The function $d_2 \colon (\overline{\mathcal{R}} - \{0\})^2 \to \mathcal{R}_0^+$, defined by $d_2(x, y) = \frac{1}{y} - \frac{1}{x}$ when $y < x$ and 0 otherwise, and its conjugate are quasi-pseudo-metrics.

Each quasi-pseudo-metric d induces a topology $\mathcal{T}_d$ generated by the base $\{B_\epsilon[x] | \, \epsilon > 0, x \in X\}$, where $\forall \epsilon > 0 \, \forall x \in X. \, B_\epsilon[x] = \{y | \, d(x, y) < \epsilon\}$.

Let $\Delta = \{(x, x) | \, x \in X\}$. The topology $\mathcal{T}_d$ is T_0 iff $\leq_d$ is a partial order and is T_1 iff $\leq_d$ is a discrete order (e.g. [MN66]). We assume all quasi-pseudo-metric spaces to satisfy the T_0 separation axiom in what follows.

The *conjugate* d^{-1} of a quasi-pseudo-metric d is defined to be the function $d^{-1}(x, y) = d(y, x)$, which is again a quasi-pseudo-metric (e.g. [FL82]). The *conjugate* of a quasi-pseudo-metric space (X, d) is the quasi-pseudo-metric space (X, d^{-1}). The *metric* d^* induced by a quasi-pseudo-metric d is defined by $d^*(x, y) = max\{d(x, y), d(y, x)\}$.

The *associated preorder* $\leq_d$ of a quasi-pseudo-metric d is defined by $x \leq_d y$ iff $d(x, y) = 0$.

A quasi-pseudo-metric space (X, d) *has a maximum (minimum)* iff its associated partial order $(X, \leq_d)$ has a maximum (minimum). (We recall that all spaces are assumed to be T_0.)

A function $f \colon (X, d) \to \mathcal{R}$ is decreasing (increasing) iff $\forall x, y \in X. \, x \leq_d y \Rightarrow f(x) \geq f(y)$ and a function $f \colon (X, d) \to \mathcal{R}$ is strictly decreasing (strictly increasing) iff $\forall x, y \in X. \, x <_d y \Rightarrow f(x) > f(y)$.

A partial order $(X, \leq)$ is *directed* when for any two elements x, y of X there exists an element $z \in X$ such that $x, y \leq z$. A quasi-pseudo-metric space is *directed* when its associated order is directed. A *join semilattice* is a partially ordered set $(X, \leq)$ such that any two points $x, y \in X$ have a supremum $x \sqcup y$ in X. A *lattice* is a partially ordered set $(X, \leq)$ such that any two elements $x, y \in X$ have a supremum $x \sqcup y$ and an infimum $x \sqcap y$ in X.

Given a quasi-pseudo-metric space (X, d) then for any point $x \in X$, we define $x\!\uparrow = \{(x_n)_{n \geq 1} | \, x_1 \geq_d x \text{ and } \forall n \geq 1. \, x_n \leq_d x_{n+1}\}$. Note that $\forall x \in X. \, x\!\uparrow \neq \emptyset$, since $x\!\uparrow$ contains the sequence with constant value x.

For any function $f \colon A \to B$ and for any set $X \subseteq A$, $f|X$ indicates the restriction of f to the set X. A *sub space* of a quasi-pseudo-metric space (X, d) is a pair $(Y, d|Y^2)$, where $Y \subseteq X$. An *extension* of a quasi-pseudo-metric space (X, d) is a quasi-pseudo-metric space (Y, d), such that (X, d) is a sub space of (Y, d).

A quasi-pseudo-metric space (X, d) satisfies a given property *hereditarily* when every sub space of the space (X, d) satisfies this property.

For any set X, we define $X^\omega = \{f \mid f{:}\mathcal{N} \to X\}$. We will consider function spaces which are obtained from a given quasi-pseudo-metric space (X, d), of the kind $(\mathcal{F}, d^\omega)$, where $\mathcal{F} \subseteq X^\omega$ and where the distance d^ω is defined by $\forall f, g \in \mathcal{F}.\, d^\omega(f, g) = \sum_n d(f(n), g(n))\frac{1}{2^n}$, on condition that the sum converges on the elements of $\mathcal{F}$. In what follows, whenever a function space is considered, we implicitly assume that this condition is satisfied.

For instance in case d is bounded, that is $\exists K \in \mathcal{R}_0^+ \, \forall x, y \in X.\, d(x, y) \le K$, the distance d^ω is defined on the entire set X^ω.

It is easy to verify that a function space $(\mathcal{F}, d^\omega)$ has the pointwise order generated from $\le_d$ as associated order, that is $\forall f, g \in \mathcal{F}.\, f \le_{d^\omega} g \Leftrightarrow \forall n.f(n) \le_d g(n)$.

For any set $A \subset \mathcal{R}^+$, a function $f \in A^\omega$ is *bounded from below (above)* iff $\exists c > 0 \, \forall n \in \mathcal{N}.\, f(n) \ge c \, (f(n) \le c)$.

3 Weightable spaces

We recall the definition of a weightable quasi-pseudo-metric space.

Definition 1 *A quasi-pseudo-metric space (X, d) is weightable iff there exists a function $w{:}\, X \to \mathcal{R}_0^+$ such that $\forall x, y \in X.\, d(x, y) + w(x) = d(y, x) + w(y)$. This equality is referred to as the weighting equality. The function w is called a weighting function, $w(x)$ is the weight of x and the quasi-pseudo-metric d is weightable by the function w. A weighted space is a triple (X, d, w) where (X, d) is a quasi-pseudo-metric space weightable by the function w.*

Examples: The quasi-pseudo-metric space $(\mathcal{R}_0^+, d_1)$ is weightable by the identity function, $w_1(x) = x$ and the space $(\overline{\mathcal{R}^+}, d_2)$ is weightable by the function $w_2(x) = \frac{1}{x}$.

We recall the example of the weightable Baire quasi-pseudo-metric discussed in [Mat92] and [Mat92a] (cf. also [Kah74] and [Wad81]).

Let $S^{\le\omega}$ be the set of countably infinite and finite sequences of elements from a given set S. Given a sequence $s \in S^{\le\omega}$, say of length $L \ge 1$, then for any natural number n such that $1 \le n \le L$, $s(n)$ denotes the n-th element of the sequence. Define the function $p{:}\, S^{\le\omega} \times S^{\le\omega} \to \mathcal{R}_0^+$ as follows,

> $\forall x, y \in S^{\le\omega}.\, p(x, y) = 2^{-\alpha}$, where $\alpha = max\{n \mid x(n) = y(n)\}$ when the sequences x and y have a common non empty initial segment and $\alpha = 0$ otherwise.

The function p is a "partial metric", the "Baire partial metric", also referred to as the "Kahn partial metric" (e.g. [Kah74] and [Mat92a]). The weightable quasi-pseudo-metric corresponding to the Baire partial metric, is the Baire quasi-pseudo-metric b, defined by $\forall x, y \in X.\, b(x, y) = p(x, y) - p(x, x)$, with weighting function $w_b(x) = p(x, x)$ (cf. [Mat92a]).

We remark that the conjugate quasi-pseudo-metric space $(\mathcal{R}_0^+, d_1^{-1})$ is not weightable.

We sketch the argument. Assume by way of contradiction that the space $(\mathcal{R}_0^+, d_1^{-1})$ is weightable by a function w. For $x \geq y \geq 0$ we obtain, by weightedness of d_1^{-1} with respect to w, that $d_1^{-1}(x, y) = x - y = w(y) - w(x)$, which implies that w is unbounded. However, for $y = 0$, we obtain that $\forall x \geq 0.\, x = w(0) - w(x)$ and thus $\forall x \geq 0.\, w(x) \leq w(0)$, which contradicts the unboundedness of w.

Since the paper focuses on weightable quasi-pseudo-metric spaces, we will in what follows solely refer to the weightable space $(\mathcal{R}_0^+, d_1)$ and not to its conjugate.

For more information on conjugates of weightable spaces we refer the reader to [Kün93]. We recall that a topological space induced by a quasi-pseudo-metric whose conjugate is weightable, need not admit any weightable quasi-pseudo-metric ([Kün93]).

We show the following fact on weightable quasi-pseudo-metric spaces.

Lemma 2 *The weighting functions of a weightable quasi-pseudo-metric space are descending.*

Proof: Let (X, d) be a weightable quasi-pseudo-metric space and let w be a weighting function for (X, d). For any two points $x, y \in X$ such that $x \leq_d y$, we have by the weighting equality that $w(x) = d(y, x) + w(y) \geq w(y)$.

□

We introduce the "descending path condition" in connection to weightable spaces.

We use the following terminology from [Kün93]. Given a set X, then a *path* in X is a finite sequence $(x_1, ..., x_n)$ of points in X. For any two points x, y of a given set X, a path from x to y is a path $p = (x_1, ..., x_n)$ in X such that $x_1 = x, x_n = y$ and $n \geq 2$. A path $p = (x_1, ..., x_n)$ is *descending* iff $\forall i \colon 1 \ldots n-1.\, x_i \geq_d x_{i+1}$.

Definition 3 *A quasi-pseudo-metric space satisfies the descending path condition (DPC) iff* $\forall x, y, z \in X.\, x \geq_d y \geq_d z \Rightarrow d(x, y) + d(y, z) = d(x, z)$.

The descending path condition intuitively expresses that the shortest distance between two points is obtained by following descending paths.

We show that weightable quasi-pseudo-metric spaces satisfy the descending path condition. An example of a non weightable quasi-pseudo-metric space which satisfies the condition is given by the space $(\mathcal{R}_0^+, d_1^{-1})$.

Lemma 4 *Weightable quasi-pseudo-metric spaces satisfy the descending path condition.*

Proof: If (X, d) is a weightable quasi-pseudo-metric space and w is a weighting function for this space, then we obtain by the weighting equality that $\forall x, y \in X. x \geq_d y \Rightarrow d(x, y) = w(y) - w(x)$. So for any three points $x, y, z \in X$ such that $x \geq_d y \geq_d z$ we have $d(x, y) + d(y, z) = d(x, z)$ since $(w(y) - w(x)) + (w(z) - w(y)) = w(z) - w(x)$.

□

We will use the following notation for the function which represents the distance from a given point. If (X, d) is a quasi-pseudo-metric space then for any point $x_0 \in X$, the function $f_{x_0}: (X, d) \to \mathcal{R}_0^+$ is defined by $\forall x \in X. f_{x_0}(x) = d(x_0, x)$.

We show that these functions are decreasing.

Lemma 5 *If (X, d) is a quasi-pseudo-metric space then $\forall x, y, z \in X. (x' \leq_d x$ and $y' \geq_d y) \Rightarrow d(x', y') \leq d(x, y)$.*

Proof: If (X, d) is a quasi-pseudo-metric space and $x, y, z \in X$ such that $x' \leq_d x$ and $y' \geq_d y$, then $d(x', y') \leq d(x', x) + d(x, y) + d(y, y') = d(x, y)$.

□

Lemma 6 *If (X, d) is a quasi-pseudo-metric space then for any point $x_0 \in X$ the function f_{x_0} is decreasing.*

Proof: Assume that (X, d) is a quasi-pseudo-metric space and let x_0 be a point of X. If x and y are two points of X such that $x \leq_d y$ then, by Lemma 5, $f_{x_0}(y) = d(x_0, y) \leq d(x_0, x) = f_{x_0}(x)$.

□

Lemma 7 *If (X, d) is a quasi-pseudo-metric space which satisfies the descending path condition and which has a maximum x_0 then the function f_{x_0} is strictly decreasing.*

Proof: Assume that (X, d) satisfies the descending path condition and has a maximum x_0. If x and y are points of X such that $x <_d y$ then, since f_{x_0} is decreasing (Lemma 6), we have $f_{x_0}(x) \geq f_{x_0}(y)$.

Assume by way of contradiction that $f_{x_0}(x) = f_{x_0}(y)$. Since $x_0 \geq_d y >_d x$, the descending path condition implies that $f_{x_0}(y) + d(y, x) = f_{x_0}(x)$. So we obtain that $d(y, x) = 0$ and thus, since the spaces are assumed to be T_0, we obtain that $x = y$, which contradicts the hypothesis $x <_d y$.

□

We show that weightable quasi-pseudo-metric spaces with a maximum essentially posses a unique weighting function.

Lemma 8 *If (X, d) is a weightable quasi-pseudo-metric space with a maximum x_0 then its weighting functions are exactly the functions $f_{x_0} + c$ where $c \geq 0$.*

Proof: Let (X, d) be a weightable space with a maximum x_0 and let w be a weighting function for the space (X, d). Then by the weighting equality we obtain that $\forall x \in X. d(x, x_0) + w(x) = d(x_0, x) + w(x_0)$. So we have that $\forall x \in X. w(x) = d(x_0, x) + w(x_0) = f_{x_0}(x) + w(x_0)$. Hence every weighting function is of the form $f_{x_0} + c$ for some constant $c \geq 0$. Note that if, for some $c \geq 0$, $f_{x_0} + c$ is a weighting function for the space (X, d), then by substracting the constant c from the weighting equality we obtain that f_{x_0} is a weighting function for (X, d). This implies that for any constant $c \geq 0$ the function $f_{x_0} + c$ is a weighting function for this space.

□

4 Upper weightable spaces

Definition 9 *A quasi-pseudo-metric d is functionally bounded iff there exists a function $f: X \to \mathcal{R}_0^+$ such that $\forall x, y \in X. d(x, y) \leq f(y)$. A quasi-pseudo-metric is functionally bounded with respect to a point $x_0 \in X$ iff the quasi-pseudo-metric is functionally bounded by the function f_{x_0}. A quasi-pseudo-metric space (X, d) is functionally bounded when d is functionally bounded.*

Comment: There is no need to introduce notions of functionally boundedness "with respect to the second argument" ("$d(x, y) \leq f(y)$") and of functionally boundedness "with respect to the first argument ("$d(x, y) \leq f(x)$") since the last property can be expressed by the statement that d^{-1} is functionally bounded.

Examples: Any bounded quasi-pseudo-metric space is functionally bounded. This is for instance the case for the Baire quasi-pseudo-metric spaces $(S^{\leq\omega}, b)$ and for the space $(\overline{\mathcal{R}^+}, d_2)$, which are bounded by 1. The space $(\mathcal{R}_0^+, d_1)$ is functionally bounded by the function w_1. The spaces $(\mathcal{R}_0^+, d_1)$ and $(\overline{\mathcal{R}^+}, d_2)$ are functionally bounded with respect to 0 and ∞ respectively.

Definition 10 *A quasi-pseudo-metric space (X, d) is upper weightable iff there exists a weighting function w for (X, d) such that the space (X, d) is functionally bounded with respect to w. In that case the space (X, d, w) is called upper weighted. The space (X, d) is upper weightable with respect to a point $x_0 \in X$ iff (X, d) is functionally* *bounded with respect to the point x_0, such that the function f_{x_0} is a weighting function for (X, d). In that case the space (X, d, w) is called upper weighted with respect to the point x_0.*

Examples: The spaces $(\mathcal{R}^+, d_1, w_1)$ and $(\mathcal{R}^+, d_2, w_2)$ are upper weighted, while the space $(\mathcal{R}^+_0, d_1, w_1)$ is upper weighted with respect to 0 and the space $(\overline{\mathcal{R}^+}, d_2, w_2)$ is upper weighted with respect to ∞.

The notion of an upper weightable space has originally been motivated by the study of the complexity spaces. These spaces have been introduced in [Sch95] as part of the development of a topological foundation for the complexity analysis of programs. The development of this foundation has been continued in [Sch95a] based on a function space construction for upper weightable spaces which we briefly discuss below.

Definition 11 *If (X, d, w) is an upper weighted space then we define the upper weighted function space generated from (X, d, w) to be the space $(\tilde{X}, d^\omega, w^\omega)$, where $\tilde{X} = \{f : \mathcal{N} \to X \mid w \circ f$ is bounded from above$\}$, $d^\omega(f, g) = \sum_{n \geq 0} d(f(n), g(n)) \frac{1}{2^n}$ and $w^\omega f = \sum_{n \geq 0} w(f(n)) \frac{1}{2^n}$.*

To verify that the definition is sound, we note that it is easy to show that the upper weightedness of the space (X, d, w) implies that the sum used in the definition of d^ω converges and also that $(\tilde{X}, d^\omega, w^\omega)$ is upper weighted.

Examples:
1) The upper weighted function space $(\tilde{\overline{\mathcal{R}^+}}, d_2^\omega, w_2^\omega)$ generated from the upper weighted space $(\overline{\mathcal{R}^+}, d_2, w_2)$ coincides with the complexity space $(\mathcal{C}, d, w)$ originally introduced in [Sch95]. We recall that the set $\mathcal{C}$ consists of all functions from $\overline{\mathcal{R}^+}^\omega$ which are bounded from below, which coincides with the set $\tilde{\overline{\mathcal{R}^+}}$.

The quasi-pseudo-metric d_2^ω on $\mathcal{C}$, referred to as "the complexity distance" in [Sch95a], is defined by:

$$\forall f, g \in \mathcal{C}.\, d_2^\omega(f, g) = \sum_{n \geq 0} \{(\frac{1}{g(n)} - \frac{1}{f(n)}) \frac{1}{2^n} \mid f(n) > g(n)\}.$$

It is easy to verify that the space $(\mathcal{C}, d_2^\omega, w_2^\omega)$ is upper weighted with respect to the function $\top$, where $\top \in \mathcal{C}$ is the function with constant value ∞. This function intuitively corresponds to the complexity function of a program which is undefined on all inputs (cf. [Sch95]).

2) The upper weighted function space generated from the upper weighted quasi-pseudo-metric space $(\mathcal{R}^+_0, d_1, w_1)$ consists of the functions which are bounded from above.

A second motivation for the study of the upper weightable spaces lies in the fact that these spaces are tightly related to the directed weightable quasi-pseudo-metric spaces. In fact the following characterization is obtained in [Sch95a]: "*A weightable space is upper weightable iff it has a directed weightable extension*".

In connection to this result, we remark that Lemma 8 implies that any weightable space with a maximum is upper weightable. Indeed, if (X, d) is a weightable space with a maximum, say x_0, then from Lemma 8 we obtain that the function f_{x_0} is a weighting function for (X, d). So $\forall x, y \in X. \, d(x, y) \leq d(x, x_0) + d(x_0, y) = f_{x_0}(y)$.

We show the more general result that any directed weightable space is upper weightable.

Assume that the quasi-pseudo-metric space (X, d) is directed and weightable via a function w. Then we have that $\forall x, y \in X \; \exists z \geq_d x, y$ and thus $w(y) - w(z) = d(z, y) \geq d(x, y)$, by weightedness and by Lemma 5, which implies that $d(x, y) \leq w(y)$.

This implies in particular that any weightable space with a directed weightable extension is upper weightable. The converse is shown in [Sch95a].

The upper weightable spaces can be interpreted to form a class of spaces which in a sense is "orthogonal" to the class of the metric spaces inside the class of the weightable spaces.

We remark that any weighted quasi-pseudo-metric space (X, d, w) satisfies the following property: $\forall x, y \in X. \, d(x, y) \leq w(y) \Leftrightarrow d(y, x) \leq w(x)$, or equivalently: $\forall x, y \in X. \, d(x, y) \geq w(y) \Leftrightarrow d(y, x) \geq w(x)$. These properties are immediate consequences of the weighting equality. The spaces which arise as "extreme cases" with respect to these equivalent properties are the upper weightable spaces, satisfying the inequality $\forall x, y \in X. \, d(x, y) \leq w(y)$ and the lower weightable spaces, satisfying the inequality $\forall x, y \in X. \, d(x, y) \geq w(y)$.

The class of lower weightable spaces is easily characterized, since the class coincides with the class of the metric spaces. Indeed, if (X, d) is lower weightable then $\forall x \in X. \, w(x) \leq d(x, x) = 0$ and thus $\forall x \in X. \, w(x) = 0$, which implies that (X, d) is a metric space. The converse holds since any metric space is lower weightable by the function with constant value 0.

So we will restrict our attention to the upper weightable spaces in what follows.

Of course, not every weightable space falls under one of these extreme cases. The Baire quasi-pseudo-metric space $(\mathcal{N}^{\leq\omega}, b, w_b)$ provides an example of a space which is neither lower nor upper weightable. Note that the space would be upper weightable iff $\forall x, y \in \mathcal{N}^{\leq\omega}. \, b(x, y) \leq w_b(x) = p(x, x)$, that is $\forall x, y \in \mathcal{N}^{\leq\omega}. \, p(x, y) \leq 2p(x, x)$. This condition is violated for the case where x is an infinite sequence and y is a finite initial segment of x. Since the space is not a metric space, the space is not lower weightable (we leave the verifications to the reader).

We focus in what follows on a characterization of a subclass of the upper weightable spaces: the weightable optimal join semilattices.

We recall that one of our motivations to consider upper weightable spaces is to continue the development of the topological foundation for Complexity Analysis discussed in [Sch95a]. Since, as indicated above, upper weightable function spaces play a central role in this study, we aim at providing a characterization of a class of upper weightable spaces which is sufficiently large to include such function spaces. This leads to the study of the weightable optimal join semilattices, which include the examples discussed above on upper weightable spaces and the function spaces they generate.

The following section discusses the class of functionally bounded directed quasi-pseudo-metric spaces which will be useful in the characterization of the weightable optimal join semilattices.

5 Functionally bounded directed spaces

We show that any functionally bounded directed space has an extension which is functionally bounded with respect to a point.

Lemma 12 *A quasi-pseudo-metric space has a maximum x_0 iff the space is functionally bounded with respect to the point x_0.*

Proof: If a quasi-pseudo-metric space (X, d) has a maximum x_0 then $\forall x, y \in X.\, d(x, y) \leq d(x, x_0) + d(x_0, y) = d(x_0, y)$ and thus the space is functionally bounded with respect to the point x_0. To show the converse, note that when a quasi-pseudo-metric space (X, d) is functionally bounded with respect to a point, say x_0, then $\forall x \in X.\, d(x, x_0) \leq d(x_0, x_0) = 0$ and thus $\forall x \in X.\, x \leq_d x_0$ or equivalently, x_0 is the maximum of the space (X, d).

□

Theorem 13 *Every functionally bounded directed quasi-pseudo-metric space has an extension which is functionally bounded with respect to a point.*

Proof: Assume that (X, d) is a directed quasi-pseudo-metric space, which is functionally bounded by the function f.

If the quasi-pseudo-metric space (X, d) has a maximum x_0 then the result holds by Lemma 12.

In case the space (X, d) does not posses a maximum, let $X_0 = X \cup \{x_0\}$ for some point $x_0 \notin X$ and define the function d_0 as follows: $\forall x, y \in X.\, d_0(x, y) = d(x, y)$, $\forall x \in X_0.\, d_0(x, x_0) = 0$ and $\forall x \in X.\, d_0(x_0, x) = \sup\{\lim_{n\to\infty} d(x_n, x) | \, (x_n)_n \in x\uparrow\}$.

We verify that the function d_0 is well defined.

Note that for any $x \in X$ and for any sequence $(x_n)_n \in x\uparrow$, the sequence $(d_0(x_n, x))_n$ is increasing by Lemma 5 and thus the sequence $(d_0(x_n, x))_n$ converges since it is bounded by $f(x)$. The fact that for each sequence $(x_n)_n$ in $x\uparrow$, the limit of the sequence $(d_0(x_n, x))_n$ is bounded by $f(x)$ guarantees that the supremum of these limits is finite.

We show that for any $x \in X$ there exists a sequence $(y_n)_n \in x\uparrow$ such that $d_0(x_0, x) = \lim_{n\to\infty} d(y_n, x)$. Indeed, since $d_0(x_0, x) = \sup\{\lim_{n\to\infty} d(x_n, x) | (x_n)_n \in x\uparrow\}$, there exists a sequence of sequences in $x\uparrow$, say $[(x^k_n)_n]_k$, such that $d(x_0, x) = \sup_k\{\lim_{n\to\infty} d_0(x^k_n, x)\}$. Since the quasi-pseudo-metric space (X, d) is directed, we can define a sequence $(y_n)_n$ where $y_1 = x^1_1$ and $\forall n \geq 1.\, y_{n+1}$ is an element such that $y_{n+1} \geq_d y_n, x^1_n, \ldots, x^n_n$. Note that $(y_n)_n \in x\uparrow$ and that $\forall k.\, \lim_{n\to\infty} d(x^k_n, x) \leq \lim_{n\to\infty} d(y_n, x)$. So $d_0(x_0, x) = \lim_{n\to\infty} d(y_n, x)$.

Next we verify that d_0 is a quasi-pseudo-metric.

The fact that $\forall x \in X_0.\, d_0(x, x) = 0$ follows since d_0 coincides with d on X and since $d_0(x_0, x_0) = 0$.

In order to verify the triangle inequality, $\forall x, y, z \in X_0.\, d_0(x, y)+d_0(y, z) \geq d_0(x, z)$, we distinguish cases.

Case 1) $x, y, z \in X$.

The result follows by the fact that d is a quasi-pseudo-metric.

Case 2) $z = x_0$.

The inequality reduces to the inequality $\forall x, y \in X.\, d_0(x, y) \geq 0$, which obviously holds. So we can assume that $z \neq x_0$ in what follows.

Case 3) $x = x_0$.

If $y = x_0$ then the triangle inequality reduces to the trivial inequality $\forall z \in X.\, d_0 (x_0, z) \geq d_0(x_0, z)$.

We verify the case where $y \neq x_0$. Let $(y_n)_n$ be a sequence in $y\uparrow$ such that $d_0(x_0, y) = \lim_{n\to\infty} d(y_n, y)$ and similarly let $(z_n)_n$ be a sequence in $z\uparrow$ such that $d_0(x_0, z) = \lim_{n\to\infty} d(z_n, z)$. Since the quasi-pseudo-metric space (X, d) is directed, we can construct a sequence $(u_n)_n$ of $y\uparrow \cap z\uparrow$ where $u_1 \geq_{d_0} y_1, z_1$ and where $\forall n \geq 1.\, u_{n+1} \geq_d u_n, y_{n+1}, z_{n+1}$. Then we have that $d_0(x_0, y) = \lim_{n\to\infty} d(u_n, y)$ and $d_0(x_0, z) = \lim_{n\to\infty} d(u_n, z)$. Since $\forall n \geq 1.\, d(u_n, y) + d(y, z) \geq d(u_n, z)$, the result follows by taking the limit of both sides.

Case 4) $x \neq x_0$ and $y = x_0$.

We need to verify that $d_0(x_0, z) \geq d_0(x, z)$; that is whether $d_0(x_0, z) \geq d(x, z)$.

Let $(x_n)_n$ be a sequence in $x\uparrow$ such that $d_0(x_0, z) = \lim_{n\to\infty} d(x_n, z)$. Then since $\forall n \geq 1.\, x_n \geq_d x$, we have that $d_0(x_0, z) = \lim_{n\to\infty} d(x_n, z) \geq d(x, z)$.

Finally we verify that the quasi-pseudo-metric d_0 is functionally bounded with respect to x_0. If x and y are two points of X, then, for the case where $x \neq x_0$, let $(x_n)_n$ be a sequence in $x\uparrow$ such that $d_0(x_0, y) = \lim_{n\to\infty} d(x_n, y)$. Then, by directedness, we can construct a sequence $(y_n)_n$ such that $\forall n \geq 1.\, y_n \geq_d x, x_n$. So we have, as above, that $\lim_{n\to\infty} d(y_n, y) = d(x_0, y)$. Then we have that $\forall n \geq 1.\, d(x, y) \leq d(y_n, y)$ and thus $d(x, y) \leq \lim_{n\to\infty} d(y_n, y) = d_0(x_0, y)$. The verification for the case where $x = x_0$ is trivial. □

6 Optimality

We consider in what follows quasi-pseudo-metric spaces for which the associated order is a join semilattice.

By Lemma 5 we obtain that for any quasi-pseudo-metric space (X, d) the following holds: $\forall x, y, z \in X. z \geq_d x \Rightarrow d(z, y) \geq d(x, y)$. In particular, any quasi-pseudo-metric space (X, d) for which the associated order is a join semilattice, satisfies the inequality $\forall x, y \in X. d(x \sqcup y, y) \geq d(x, y)$. Similarly we obtain that any quasi-pseudo-metric space (X, d) for which the associated order is a lattice satisfies the inequality: $\forall x, y \in X. d(x, x \sqcap y) \geq d(x, y)$, in addition to the previous one.

We will consider the "optimal" cases where $d(x \sqcup y, y) = d(x, y)$ and $d(x, x \sqcap y) = d(x, y)$.

Definition 14 *A quasi-pseudo-metric (join semi)lattice is a quasi-pseudo-metric space for which the associated order is a (join semi)lattice. A quasi-pseudo-metric join semilattice* (X, d) *is optimal iff* $\forall x, y \in X. d(x \sqcup y, y) = d(x, y)$*. The notion of an optimal quasi-pseudo-metric lattice is defined in a similar way. For the sake of brevity, we refer to optimal quasi-pseudo-metric (join semi)lattices in what follows as optimal (join semi)lattices, in which case the quasi-pseudo-metric space is referred to as the "underlying quasi-pseudo-metric space".*

Examples: Any quasi-pseudo-metric space for which the associated partial order is linear is an optimal lattice.

Any upper weighted function space generated from a given optimal upper weighted (join semi)lattice is optimal, where the operations meet and join are defined by point wise extension of the operations on the original space. We leave the straightforward verifications to the reader. Some specific examples are the complexity space $(\mathcal{C}, d)$ and the upper weighted function space generated from the upper weighted space $(\mathcal{R}_0^+, d_1, w_1)$.

Of course, not every quasi-pseudo-metric join semilattice (X, d) is optimal. A counterexample is given by the space (X, d), where $X = \{(0, 1), (1, 3), (4, 1)\}$ and where $\forall (x_1, y_1), (x_2, y_2) \in X. d((x_1, y_1), (x_2, y_2)) = d_1(x_1, x_2) + d_1(y_1, y_2)$.

This space is upper weightable via the function $w((x_1, y_1)) = x_1 + y_1$. So the condition of upper weightability does not imply optimality.

Theorem 15 *An optimal join semilattice is weightable iff the underlying quasi-pseudo-metric space is functionally bounded and satisfies the descending path condition.*

Proof: Let (X, d) be an optimal join semilattice.

If (X, d) is weightable then, since the space is directed, it is upper weightable and thus in particular functionally bounded. By Lemma 4, (X, d) satisfies the descending path condition.

To show the converse, assume that (X, d) is functionally bounded and satisfies the descending path condition.

By Theorem 13, there exists an extension (X_0, d_0) of (X, d) which is functionally bounded with respect to a point x_0, where $X_0 = X \cup \{x_0\}$ in case the space (X, d) does not have a maximum and $X_0 = X$ otherwise (we use the notation introduced in the proof of Theorem 13).

Since weightability is a hereditary property, in order to show that the space (X, d) is weightable it suffices to show that the space (X_0, d_0) is weightable. We will verify that the space (X_0, d_0) is weightable by the weighting function w_0, defined by $\forall x \in X_0.\, w_0(x) = d_0(x_0, x)$.

We show that $\forall x, y \in X_0.\, x \geq_{d_0} y \Rightarrow d_0(x_0, x) + d_0(x, y) = d_0(x_0, y)$. Equivalently we need to show that $\forall x, y \in X_0.\, x \geq_{d_0} y \Rightarrow d_0(x, y) = w_0(y) - w_0(x)$. We distinguish cases.

The verifications of the cases where $x = x_0$ or $y = x_0$ are straightforward.

If $x, y \in X$ such that $x \geq_{d_0} y$, then, since d_0 and d coincide on X, we have $x \geq_d y$ and thus for some sequence $(x_n)_n \in x\uparrow$ we have that $d_0(x_0, x) + d_0(x, y) = \lim_{n\to\infty} d(x_n, x) + d_0(x, y) = \lim_{n\to\infty}(d(x_n, x) + d(x, y)) = \lim_{n\to\infty} d(x_n, y)$, where the last equality follows by the fact that (X, d) satisfies the descending path condition. So to obtain the result it suffices to show that $\lim_{n\to\infty} d(x_n, y) = d_0(x_0, y)$. Since $(x_n)_n \in x\uparrow$ and $x \geq_d y$, we have that $(x_n)_n \in y\uparrow$ and thus the inequality $\lim_{n\to\infty} d(x_n, y) \leq d_0(x_0, y)$ holds.

We assume by way of contradiction that $\lim_{n\to\infty} d(x_n, y) < d_0(x_0, y)$. Then there exists a sequence $(y_n)_n \in y\uparrow$ such that $\lim_{n\to\infty} d(y_n, y) > \lim_{n\to\infty} d(x_n, y)$. However, for the sequence $(y_n \sqcup x)_n \in x\uparrow$, we obtain that $\lim_{n\to\infty} d(y_n \sqcup x, y) > \lim_{n\to\infty} d(x_n, y)$. Since $d(y_n \sqcup x, y) = d(y_n \sqcup x, x) + d(x, y)$ and $d(x_n, y) = d(x_n, x) + d(x, y)$ by the descending path condition, we obtain that $\lim_{n\to\infty} d(y_n \sqcup x, x) > \lim_{n\to\infty} d(x_n, x) = d_0(x_0, x)$, which yields a contradiction.

It is straightforward to verify that (X_0, d_0) is still an optimal join semilattice. So we have that $\forall x, y \in X_0.\, d_0(x, y) = d_0(x \sqcup y, x)$ and $d_0(y, x) = d_0(y \sqcup x, x)$. Since we have shown that $\forall x, y \in X_0.\, x \geq_{d_0} y \Rightarrow d_0(x, y) = w_0(y) - w_0(x)$, we obtain that $\forall x, y \in X_0.\, d_0(x, y) - d_0(y, x) = d_0(x \sqcup y, y) - d_0(x \sqcup y, x) = (w_0(y) - w_0(x \sqcup y)) - (w_0(x) - w_0(x \sqcup y)) = w_0(y) - w_0(x)$. So (X_0, d_0) is weighted with respect to w_0 and hence (X, d) is weighted with respect to the function $w_0|X$.

□

The following result provides some information on weighted optimal lattices.

Proposition 16 *A weighted optimal lattice $(X, d, w, \sqcup, \sqcap)$ satisfies the following laws:*

$$\forall x, y \in X.\, w(x \sqcap y) + w(x \sqcup y) = w(x) + w(y) \text{ and}$$

$$\forall x, y \in X.\, w(x \sqcap y) - w(x \sqcup y) = d(x, y) + d(y, x).$$

Proof: By weightedness we have that $\forall x, y \in X.\, d(x,y) - d(y,x) = w(y) - w(x)$, which by the optimality of the quasi-pseudo-metric lattice is equivalent to $\forall x, y \in X.\, d(x \sqcup y, y) - d(y, x \sqcap y) = w(y) - w(x)$. Since $\forall x, y \in X.\, x \sqcup y \geq_d y$ and $y \geq_d x \sqcap y$ we obtain, again by weightedness, that $d(x \sqcup y, y) = w(y) - w(x \sqcup y)$ and $d(y, x \sqcap y) = w(x \sqcap y) - w(y)$. Hence $\forall x, y \in X.\, (w(y) - w(x \sqcup y)) - (w(x \sqcap y) - w(y)) = w(y) - w(x)$ and thus $\forall x, y \in X.\, w(x) + w(y) = w(x \sqcup y) + w(x \sqcap y)$. To verify the second equality, note that $\forall x, y \in X.\, d(x,y) + d(y,x) = d(x \sqcup y, y) + d(y, x \sqcap y) = w(x \sqcap y) - w(x \sqcup y)$.

□

We conclude the paper with an alternative characterization of weightable optimal join semilattices.

Note that for optimal weightable join semilattices (X, d), say with a weighting function w, we have that $\forall x, y \in X.\, w(x \sqcup y) = w(y) - d(x, y)$. Since these spaces are directed they are in particular upper weightable.

Conversely, an upper weightable space allows one to define the following equation with unknown z in X: $\forall x, y \in X.\, w(z) = w(y) - d(x, y)$, based on the fact that the expression $w(y) - d(x, y)$ is guaranteed to be positive.

The solution intuitively corresponds to an "optimal supremum" of the points x and y. This intuition lies at the basis of the following definition and theorem.

Definition 17 *A weighting function is upper solvable iff* $\forall x, y \in X\, \exists z \in X.\, z \geq_d x, y$ *and* $w(z) = w(y) - d(x, y)$. *A weightable space is upper solvable iff it has an upper solvable weighting function.*

We remark that any upper solvable weightable space is upper weightable.

Lemma 18 *If* (X, d, w) *is a weighted quasi-pseudo-metric space then*

$$(*)\ \forall x, y, z \in X.\, z \geq_d x, y \Rightarrow w(z) \leq_d w(y) - d(x, y).$$

Proof: Note that for $x, y, z \in X$ such that $z \geq_d x, y$ we have that $d(x, y) \leq_d d(z, y)$ (by Lemma 5). By the fact that $z \geq_d y$ and by the weighting equality, we have that $d(z, y) = w(y) - w(z)$ and thus $d(x, y) \leq w(y) - w(z)$, which is equivalent to the desired inequality.

□

Theorem 19 *A weightable space* (X, d, w) *is an optimal join semilattice iff the space is upper solvable and satisfies the following condition:*

$$\forall x, y, z_1, z_2 \in X.\, (z_1 \geq_d x, y \text{ and } z_2 \geq_d x, y) \Rightarrow (\exists z \in X.\, z_1, z_2 \geq_d z \geq_d x, y).$$

Proof: Assume that (X, d) is a weightable quasi-pseudo-metric space which has an upper solvable weighting function w and which satisfies $\forall x, y, z_1, z_2 \in X. (z_1 \geq_d x, y \text{ and } z_2 \geq_d x, y) \Rightarrow (\exists z \in X. z_1, z_2 \geq_d z \geq_d x, y)$.

Next we show that any equation of the form $w(z) = w(y) - d(x, y)$ where $x, y \in X$ and $z \geq_d x, y$ has a unique solution. Let $x, y \in X$ and assume that z_1 and z_2 are points in X such that $z_1 \geq_d x, y$ and $z_2 \geq_d x, y$, where $w(z_1) = w(z_2) = w(y) - d(x, y)$.

Note that by hypothesis there exists an element z such that $z_1, z_2 \geq_d z \geq_d x, y$. By $(*)$ (Lemma 18) and by the fact that $z \geq_d x, y$, we obtain that $w(z) \leq w(y) - d(x, y)$.

Since $z_1, z_2 \geq_d z$ we also have by Lemma 2 that $w(y) - d(x, y) = w(z_1) = w(z_2) \leq w(z)$. Thus $w(z) = w(y) - d(x, y)$ and thus $w(z_1) = w(z_2) = w(z)$.

Since $z_1, z_2 \geq_d z$, by the weighting equality we have that $d(z_1, z) = w(z) - w(z_1)$ and $d(z_2, z) = w(z) - w(z_2)$. Thus $d(z_1, z) = d(z_2, z) = 0$, that is $z_1 \leq_d z$ and $z_2 \leq_d z$, and thus $z_1 = z_2 = z$.

Given two points $x, y \in X$, let z_0 be the unique solution to the equation $w(z) = w(y) - d(x, y)$, where $z_0 \geq_d x, y$. We show that z_0 is the supremum of x and y with respect to the associated order. Assume that $z_1 \in X$ is such that $z_1 \geq_d x, y$. Then we have that $\exists z \in X. z_0, z_1 \geq_d z \geq_d x, y$. Since $z \leq_d z_0$ we have that $w(z_0) \leq w(z)$. However since $z \geq x, y$, by $(*)$ we obtain that $w(z) \leq w(y) - d(x, y)$ and thus $w(z) = w(y) - d(x, y)$. Since $z \geq_d x, y$, we obtain by uniqueness of the solution that $z = z_0$ and thus, since $z \leq_d z_1$, we have that $z_0 \leq_d z_1$.

We leave the straightforward verifications of the converse to the reader.

□

Finally we remark that problem 10 of [KV93] "Develop a notion of a weighted quasi-uniform space" can not be solved via an axiomatization in terms of the elements of the quasi-uniform space which would guarantee the weightability of all quasi-pseudo-metrics which induces the quasi-uniformity (for the case of quasi-uniformities with a countable base). This is easily shown by the following counterexample[2].

Consider the weightable quasi-pseudo-metric space $(\mathcal{R}_0^+, d_1)$. The quasi-pseudo-metric d' defined by $d'(x, y) = min(d_1(x, y), 1)$ induces the same quasi-uniformity as d_1 but does not satisfy the descending path condition and hence is not weightable.

Since the counterexample provides a space which belongs to the weightable optimal join semilattices, the same remark holds for this class as well as for the class of the upper weightable spaces.

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[2] After an example suggested by W. Lindgren.

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