The quasi-metric of complexity convergence

Salvador Romaguera*, Michel Schellekens

Abstract

For any weightable quasi-metric space (X, d) having a maximum with respect to the associated order $\leq_d$, the notion of the quasi-metric of complexity convergence on the the function space (equivalently, the space of sequences) X^ω, is introduced and studied.We observe that its induced quasi-uniformity is finer than the quasi-uniformity of pointwise convergence and weaker than the quasi-uniformity of uniform convergence. We show that it coincides with the quasi-uniformity of pointwise convergence if and only if the quasi-metric space (X, d) is bounded and it coincides with the quasi-uniformity of uniform convergence if and only if X is a singleton.We also investigate completeness of the quasi-metric of complexity convergence. Finally, we obtain versions of the celebrated Grothendieck theorem in this context.

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1 Introduction and preliminaries

Throughout this paper the letters $\mathbb{R}$, $\mathbb{R}^+$, ω and $\mathbb{N}$ will denote the set of all real numbers, of all nonnegative real numbers, of all nonnegative integer numbers and of all positive integer numbers, respectively. Our basic references for quasi-uniform and quasi-metric spaces are [2] and [4].

In our context a quasi-metric on a (nonempty) set X is a nonnegative real-valued function d on $X \times X$ such that for all $x, y, z \in X$: (i) $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$, and (ii) $d(x, y) \leq d(x, z) + d(z, y)$.

We will also consider extended quasi-metrics. They satisfy the usual axioms for a quasi-metric, except that we allow $d(x, y) = +\infty$.

If d is a(n extended) quasi-metric on X, then the function d^{-1} defined on $X \times X$ by $d^{-1}(x, y) = d(y, x)$ is also a(n extended) quasi-metric on X called the conjugate of d, and the function d^s defined on $X \times X$ by $d^s(x, y) = \max\{d(x, y), d(y, x)\}$ is a(n extended) metric on X.

A(n extended) quasi-metric space is a pair (X, d) such that X is a (non empty) set and d is a(n extended) quasi-metric on X.

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Each extended quasi-metric d on X induces a T_0 topology $\mathcal{T}(d)$ on X which has as a base the family of open balls $\{S_d(x,r) \mid x \in X,\ r > 0\}$, where $S_d(x,r) = \{y \in X \mid d(x,y) < r\}$ for all $x \in X$ and $r > 0$.

Each extended quasi-metric d on X induces a quasi-uniformity $\mathcal{U}_d$ on X (see [2, p. 3]) which has as a base the family of sets of the form $\{(x,y) : d(x,y) < 2^{-n}\}$, $n \in \mathbb{N}$.

According to [2] a quasi-uniform space $(X,\mathcal{U})$ is called bicomplete if the uniform space $(X,\mathcal{U}^s)$ is complete, where $\mathcal{U}^s$ is the coarsest uniformity on X finer than $\mathcal{U}$ and $\mathcal{U}^{-1}$. A(n extended) quasi-metric space (X,d) is said to be bicomplete if $(X,\mathcal{U}_d)$ is a bicomplete quasi-uniform space, equivalently if (X,d^s) is a complete (extended) metric space.

In [14] and [15], Smyth presented a topological framework for denotational semantic based on the theory of complete (and totally bounded) quasi-uniform and quasi-metric spaces. Sünderhauf continued this work in the setting of topological quasi-uniform spaces [17]. Künzi characterized in [4] both Smyth completable and Smyth complete quasi-uniform spaces in terms of left K-Cauchy filters as discussed in [7].

A quasi-uniform space $(X,\mathcal{U})$ is Smyth completable if and only if every left K-Cauchy filter on $(X,\mathcal{U})$ is a Cauchy filter on the uniform space $(X,\mathcal{U}^s)$ [4], where a filter $\mathcal{F}$ on $(X,\mathcal{U})$ is left K-Cauchy provided that for each $U \in \mathcal{U}$ there is an $F \in \mathcal{F}$ such that $U(x) \in \mathcal{F}$ for all $x \in F$ [7], [8]. A quasi-uniform space $(X,\mathcal{U})$ is Smyth complete if and only if every left K-Cauchy filter on $(X,\mathcal{U})$ is convergent with respect to the uniform topology $T(\mathcal{U}^s)$ [4]. Therefore, every bicomplete Smyth completable quasi-uniform space is Smyth complete. A quasi-metric space (X,d) is Smyth completable (resp. Smyth complete) if $(X,\mathcal{U}_d)$ is a Smyth completable (resp. Smyth complete) quasi-uniform space.

The weightable quasi-metric spaces, or the equivalent partial metric spaces, were introduced by Matthews [6], as a part of the study of denotational semantics of dataflow networks. Let us recall that a quasi-metric space (X,d) is called weightable if there is a function $w : X \to \mathbb{R}^+$, such that $w(x) + d(x,y) = w(y) + d(y,x)$ for all $x,y \in X$. The function w is said to be a weighting function for (X,d). It was proved in [4] that every weightable quasi-metric space is Smyth completable. Hence, every weightable bicomplete quasi-metric space is Smyth complete.

As usual, the associated order $\leq_d$ of a quasi-metric space (X,d) is defined by $x \leq_d y \Leftrightarrow d(x,y) = 0$. A quasi-metric space has a maximum if the associated order has a maximum (that is, for some $x_0 \in X$, $x \leq_d x_0$ for each $x \in X$). Note that if a quasi-metric space has a maximum, it is unique.

The theory of Denotational Semantics originated through the work by Scott and Strachey (see [16]) and focuses on the development of models for programming languages. A well known example which illustrates the need for the development of formal models is the programming language Algol60, for which the specifications in natural language allowed for different interpretations. This allowed for different implementations of this language and hence resulted in a lack of portability.

Semantics models allow one to formally show that (well designed) programming languages are free of such ambiguities by formally specifying the meaning of each part of the language. Tipically, the denotation (i.e. the “meaning”) of a program is represented via a particular fixed point of a continuous functional on a model referred to as a "Scott domain". An important point in this context is that the fixed point can be computed as the supremum of an increasing sequence, obtained by iterating the functional on the minimum $\perp$ of the Scott domain. The applications of Denotational Semantics include aiding language design, establishing standards for implementation, reasoning about programs and generating compilers.

Recent developments in Denotational Semantics include the study of semantic frameworks for complexity. The complexity (quasi-metric) space (C, d_C) was introduced in [11] to study complexity analysis of programs (see Example 3, below). It was proved in [11] that every complexity space is weightable. Applications of this theory to the complexity analysis of Divide & Conquer algorithms have also been discussed in [11]. In particular, it has been shown that each Divide & Conquer algorithm gives rise to a functional which is a contraction map on a complexity space. The complexity of such an algorithm then is represented via the fixed point of the map obtained by the Banach Theorem. In this context, the fixed point is obtained by taking the limit of a sequence, obtained by iterating the functional on the maximum $\top$ of the complexity space We remark that the fact that a maximum is used rather than a minimum, as in traditional domain theory, can be avoided by using the dual complexity space as discussed below.

The dual complexity space (C^*, d_{C^*}) is introduced in [9] (see Example 4, below). Via the analysis of its dual, several quasi-metric properties of the complexity space are studied in [9]. A motivation for the use of the dual instead of the original complexity space is the fact that the dual is mathematically somewhat more appealing. Consequently, the presentation of the proofs becomes somewhat more elegant. Furthermore, it is possible to carry out the complexity analysis of algorithms based on the dual complexity space. In fact, the dual complexity space has the advantage that it respects the interpretation usually given to the minimum $\perp$ in semantic domains (see [9, Section 4]).

In this paper we introduce and study a general notion of complexity space from a convergence point of view. In particular, a complexity space will be a certain quasi-metric subspace $(\mathcal{C}_{x_0}, d_{\mathcal{C}_{x_0}})$ of the function space X^ω, where (X, d) is any weightable quasi-metric space with a maximum x_0. The quasi-metric $d_{\mathcal{C}_{x_0}}$ will be called the quasi-metric of complexity convergence and the quasi-uniformity induced by the quasi-metric $d_{\mathcal{C}_{x_0}}$ will be called the quasi-uniformity of complexity convergence. This quasi-uniformity is weaker than the quasi-uniformity of uniform convergence and finer than the quasi-uniformity of pointwise convergence on X^ω. We show that it coincides with the quasi-uniformity of pointwise convergence if and only if d is a bounded quasi-metric on X and that it coincides with the quasi-uniformity of uniform convergence only in the case that X is a singleton. Actually, these results will be obtained in a very general setting. We also prove that Smyth completeness is preserved by the quasi-uniformity of complexity convergence and, finally, we obtain versions of the celebrated Grothendieck theorem in this context. Several illustrative examples are also given.

2 The quasi-metric of complexity convergence

Let (X,d) be a quasi-metric space. Fix $x \in X$ and define

$$\mathcal{B}_x = \{f \in X^\omega \mid \sum_{n=0}^{\infty} 2^{-n} d^s(x, f(n)) < +\infty\}.$$

and

$$d_{\mathcal{B}_x}(f,g) = \sum_{n=0}^{\infty} 2^{-n} d(f(n), g(n))$$

for all $f, g \in \mathcal{B}_x$.

Note that for each $x \in X$, $\mathcal{B}_x \neq \emptyset$, because the function $f_x : \omega \to X$ defined by $f_x(n) = x$ for all $n \in \omega$, is in $\mathcal{B}_x$. Moreover, $d_{\mathcal{B}_x}$ is a quasi-metric on $\mathcal{B}_x$ (in fact, note, in particular, that for each $f, g \in \mathcal{B}_x$, one has: $\sum_{n=0}^{\infty} 2^{-n} d^s(f(n), g(n)) \leq \sum_{n=0}^{\infty} 2^{-n} d^s(f(n), x) + \sum_{n=0}^{\infty} 2^{-n} d^s(x, g(n)) < +\infty$.)

Then, by $\mathcal{U}_{d_{\mathcal{B}_x}}$ we will denote the quasi-uniformity induced on $\mathcal{B}_x$ by $d_{\mathcal{B}_x}$.

Remark 1. The definition of the space $\mathcal{B}_x$ may seems somewhat surprising at first because it could be considered more natural to define this space as $\{f \in X^\omega \mid \sum_{n=0}^{\infty} 2^{-n} d(x, f(n)) < +\infty\}$. However the following simple example justifies our selection:

Consider the quasi-metric space $(\mathbb{R}, u)$, where u is the quasi-metric defined on $\mathbb{R}$ by $u(x,y) = \max\{y - x, 0\}$. Define $f : \omega \to \mathbb{R}$ by $f(n) = -2^n$ for all $n \in \omega$, and $g : \omega \to \mathbb{R}$ by $g(n) = 0$ for all $n \in \omega$. Then $\sum_{n=0}^{\infty} 2^{-n} u(0, f(n)) = 0$, $\sum_{n=0}^{\infty} 2^{-n} u(0, g(n)) = 0$, but $\sum_{n=0}^{\infty} 2^{-n} u(f(n), g(n)) = +\infty$. So, $u_{\mathcal{B}_0}$ would not be a quasi-metric.

Remark 2. Note that if (X,d) is a quasi-metric space and $x \in X$, then for every $f, g \in \mathcal{B}_x$, we have

$$(d_{\mathcal{B}_x})^s(f,g) \leq \sum_{n=0}^{\infty} 2^{-n} d^s(f(n), g(n)) \leq d_{\mathcal{B}_x}(f,g) + d_{\mathcal{B}_x}(g,f).$$

It is interesting to note that the inequality "$\leq$ " cannot be replaced by equality in the preceding relations. For instance, consider the quasi-metric space $(\mathbb{R}, u)$ of Remark 1 and put $x = 0$. Define two functions f and g on ω by $f(n) = 0$ for n even and $f(n) = 1$ for n odd, and $g(n) = 1$ for n even and $g(n) = 0$ for n odd. Clearly, both f and g are in $\mathcal{B}_0$. Furthermore $u_{\mathcal{B}_0}(f,g) = 4/3$, $u_{\mathcal{B}_0}(g,f) = 2/3$, so $(u_{\mathcal{B}_0})^s(f,g) = 4/3$, and $\sum_{n=0}^{\infty} 2^{-n} u^s(f(n), g(n)) = 2$. For the other inequality, consider the same functions f and g, $x = 0$ and take d as the Euclidean metric u^s.

Definition 1. Let (X,d) be a weightable quasi-metric space which has a maximum x_0. Then, the quasi-metric space $(\mathcal{B}_{x_0}, d_{\mathcal{B}_{x_0}})$ is called *the complexity space* (of (X,d)), the quasi-metric $d_{\mathcal{C}_{x_0}}$ is called *the quasi-metric of complexity convergence* and the quasi-uniformity $\mathcal{U}_{d_{\mathcal{B}_{x_0}}}$ induced by $d_{\mathcal{B}_{x_0}}$ is called *the quasi-uniformity of complexity convergence.*

The preceding definition is motivated by the fact that several interesting quasi-metric spaces which appear in Theoretical Computer Science are weightable and they have a maximum. Furthermore, these are sufficient conditions to obtain satisfactory results on completeness. In particular, we shall prove that, if (X, d) is a Smyth complete weightable quasi-metric space with a maximum x_0, then, the complexity space $(\mathcal{B}_{x_0}, d_{\mathcal{B}_{x_0}})$ is Smyth complete, while both the quasi-uniformity of uniform convergence and the quasi-uniformity of pointwise convergence are not Smyth complete on $\mathcal{B}_{x_0}$, in general (see Theorem 3, Remark 5 and Example 6, below).

Next we give some illustrative examples of weightable quasi-metric spaces with a maximum.

Given a quasi-metric space (X, d) and a subset A of X, the restriction of d to A will be also denoted by d if no confusion arises.

Example 1. Let u be the quasi-metric defined in Remark 1. It is well known that although $(\mathbb{R},u)$ is not weightable, $(\mathbb{R}^+, u)$ is a weightable quasi-metric space with weighting function w defined by $w(x) = x$ for all $x \in \mathbb{R}^+$. Moreover, 0 is the maximum of $(\mathbb{R}^+, u)$.

Example 2. The function u_{-1} defined on $(0, +\infty] \times (0, +\infty]$ by $u_{-1}(x, y) = \max\{\frac{1}{y} - \frac{1}{x}, 0\}$ is a quasi-metric on $(0, +\infty]$, where we adopt the convention that $\frac{1}{\infty} = 0$. Furthermore, $((0, +\infty], u_{-1})$ is weightable with weighting function w defined by $w(x) = \frac{1}{x}$ for all $x \in (0, +\infty]$ (see [12]). Clearly, ∞ is the maximum of $((0, +\infty], u_{-1})$. Note that this space is isometric to $(\mathbb{R}^+, u)$.

Example 3 (see [11]). The complexity space (of $((0, +\infty], u_{-1})$) is the pair (C, d_C), where $C = \{f : \omega \to (0, +\infty] \mid \sum_{n=0}^{\infty} 2^{-n} \frac{1}{f(n)} < +\infty\}$ and d_C is the quasi-metric defined on C by $d_C(f, g) = \sum_{n=0}^{\infty} 2^{-n} u_{-1}(f(n), g(n))$, whenever $f, g \in C$. The complexity space (C, d_C) is a weightable quasi-metric space with weighting function w defined by on C by $w(f) = \sum_{n=0}^{\infty} 2^{-n} \frac{1}{f(n)} < +\infty$, and it has the constant function f_∞ defined on ω by $f_\infty(n) = +\infty$ for all $n \in \omega$, as its maximum.

Example 4 (see [9]). The dual complexity space is the pair (C^*, d_{C^*}), where $C^* = \{f : \omega \to \mathbb{R}^+ \mid \sum_{n=0}^{\infty} 2^{-n} f(n) < +\infty\}$ and d_{C^*} is the quasi-metric defined on C^* by $d_{C^*}(f, g) = \sum_{n=0}^{\infty} 2^{-n} u(f(n), g(n))$, whenever $f, g \in C^*$. The dual complexity space (C^*, d_{C^*}) is a weightable quasi-metric space with weighting function w defined on C^* by $w(f) = \sum_{n=0}^{\infty} 2^{-n} f(n)$, and it has the constant function f_0 defined on ω by $f_0(n) = 0$ for all $n \in \omega$, as its maximum. Note that, following Definition 1, (C^*, d_{C^*}) is the complexity space of $(\mathbb{R}^+, u)$.

Let us recall that a quasi-metric space (X, d) is totally bounded provided that

(X, d^s) is a totally bounded metric space.

Example 5. It is shown in [15] that every Scott domain can be represented by a totally bounded quasi-metric space (see [15] fore the notion of a Scott domain). Recently, it was proved in [13] that every Scott domain can be represented by a quasi-metric space whose conjugate is weightable and possess a maximum. Thus, we also obtain a large class of quasi-metric spaces which, in view of Smyth's paper [15], are appropriate domains of computation and whose conjugates are weightable quasi-metric spaces having a maximum.

Remark 3. Let (X, d) be a weightable quasi-metric space with a maximum x_0. Put

$$\mathcal{C}_{x_0} = \{f \in \mathcal{B}_{x_0} \mid \sum_{n=0}^{\infty} 2^{-n} d(x_0, f(n)) < +\infty\}.$$

Choose any $f \in \mathcal{B}_{x_0}$. Then, $d^s(x_0, f(n)) = d(x_0, f(n))$ for all $n \in \omega$. So, $f \in \mathcal{C}_{x_0}$, and, thus, $\mathcal{C}_{x_0} = \mathcal{B}_{x_0}$.

For this reason, and according to the usual notation for complexity spaces [11], [9], [10], the complexity space $(\mathcal{B}_{x_0}, d_{\mathcal{B}_{x_0}})$ (generated by a weightable quasi-metric space (X, d) having a maximum x_0) will be denoted in the following by $(\mathcal{C}_{x_0}, d_{\mathcal{C}_{x_0}})$ and the quasi-uniformity of complexity convergence will be denoted by $\mathcal{U}_{d_{\mathcal{C}_{x_0}}}$.

Note that, according to the above terminology, the space (C, d_C) of Example 3 is exactly the complexity space $(\mathcal{C}_\infty, u_{\mathcal{C}_\infty})$ of the space $((0, +\infty], u_{-1})$ in Example 2, while the space (C^*, d_{C^*}) of Example 4 is exactly the complexity space $(\mathcal{C}_0, u_{\mathcal{C}_0})$ of the space $(\mathbb{R}^+, u)$ in Example 1.

Given a quasi-uniform space $(X, \mathcal{U})$, we denote by $\mathcal{U}_X$ and $\mathcal{U}_P$ the quasi-uniformity of uniform convergence of $\mathcal{U}$ and the quasi-uniformity of pointwise convergence (of $\mathcal{U}$) on X^ω, respectively (see, for instance, [5]).

If (X, d) is a quasi-metric space, then the quasi-uniformity of uniform convergence on X^ω (of $\mathcal{U}_d$) is the quasi-uniformity $\mathcal{U}_{d_X}$ induced on X^ω by the extended quasi-metric (of uniform convergence) d_X defined on $X^\omega \times X^\omega$, by

$$d_X(f, g) = \sup\{d(f(n), g(n)) \mid n \in \omega\},$$

while the quasi-uniformity of pointwise convergence on X^ω (of $\mathcal{U}_d$) is the quasi-uniformity $\mathcal{U}_{d_P}$ induced on X^ω by the quasi-metric (of pointwise convergence) d_P defined on $X^\omega \times X^\omega$ by

$$d_P(f, g) = \sum_{n=0}^{\infty} 2^{-n} \min\{1, d(f(n), g(n))\}.$$

The proof of the next result follows immediately from the notions given above and it is omitted.

Proposition 1. *Let (X,d) be a quasi-metric space and let $x \in X$. Then, $\mathcal{U}_{d_P} \subseteq \mathcal{U}_{d_{\mathcal{B}_x}} \subseteq \mathcal{U}_{d_X}$ on $\mathcal{B}_x$.*

It then seems interesting to characterize when $\mathcal{U}_{d_{\mathcal{B}_x}} = \mathcal{U}_{d_X}$ and when $\mathcal{U}_{d_{\mathcal{B}_x}} = \mathcal{U}_{d_P}$. In this direction we have the following results.

Proposition 2. *Let (X,d) be a quasi-metric space and let $x \in X$. Then, $\mathcal{U}_{d_{\mathcal{B}_x}} = \mathcal{U}_{d_X}$ on $\mathcal{B}_x$ if and only if $X = \{x\}$.*

Proof. Since the part 'if' is obvious we shall show the part 'only if'. To this end, suppose that the topology $\mathcal{T}(\mathcal{U}_{d_{\mathcal{B}_x}})$ generated by $\mathcal{U}_{d_{\mathcal{B}_x}}$ coincides with the topology $\mathcal{T}(\mathcal{U}_{d_X})$ generated by $\mathcal{U}_{d_X}$ on $\mathcal{B}_x$, but X has at least a point y different from x. Suppose that $d(x,y) = \delta > 0$. Define a function $f : \omega \to X$ by $f(n) = x$ for all $n \in \omega$, and construct a sequence $(f_k)_{k\in\mathbb{N}}$ of functions from ω to X by $f_k(n) = x$ if $n < k$, and $f_k(n) = y$, otherwise. Clearly $f \in \mathcal{B}_x$ and $f_k \in \mathcal{B}_x$ for all $k \in \mathbb{N}$. Since $d_X(f, f_k) = \delta$ for all $k \in \mathbb{N}$, the sequence $(f_k)_{k\in\mathbb{N}}$ does not converges to f with respect to $\mathcal{T}(\mathcal{U}_{d_X})$. On the other hand, given $\varepsilon > 0$ there is $k_\varepsilon \in \mathbb{N}$ such that $\delta/2^{k_\varepsilon - 1} < \varepsilon$. Thus, for each $k \geq k_\varepsilon$, we obtain

$$\sum_{n=0}^{\infty} 2^{-n} d(f(n), f_k(n)) = \sum_{n=k}^{\infty} 2^{-n} d(x,y) = \frac{\delta}{2^{k-1}} < \varepsilon.$$

So, $(f_k)_{k\in\mathbb{N}}$ converges to f with respect to $\mathcal{T}(\mathcal{U}_{d_{\mathcal{B}_x}})$, a contradiction. Finally, if $d(x,y) = 0$, we have $d(y,x) > 0$ and a trivial modification of the technique used above completes the proof.

Proposition 3. *Let (X,d) be a quasi-metric space and let $x \in X$. Then, $\mathcal{U}_{d_{\mathcal{B}_x}} = \mathcal{U}_{d_P}$ on $\mathcal{B}_x$ if and only if d is a bounded quasi-metric on X.*

Proof. Suppose that $\mathcal{U}_{d_{\mathcal{B}_x}} = \mathcal{U}_{d_P}$ on $\mathcal{B}_x$. Then $(\mathcal{U}_{d_{\mathcal{B}_x}})^s = (\mathcal{U}_{d_P})^s$ on $\mathcal{B}_x$. We shall show that there is an $M > 0$ such that $d^s(x,y) \leq M$ for all $y \in X$. Assume the contrary. Then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of points in X such that $d^s(x, x_n) > 2^n$ for all $n \in \mathbb{N}$. Define a function $f : \omega \to X$ by $f(n) = x$ for all $n \in \mathbb{N}$, and construct a sequence $(f_k)_{k\in\mathbb{N}}$ of functions from ω to X by $f_k(n) = x$ if $n < k$ and $f_k(n) = x_k$ otherwise. Clearly $f \in \mathcal{B}_x$ and $f_k \in \mathcal{B}_x$ for all $k \in \mathbb{N}$. Moreover, $(f_k)_{k\in\mathbb{N}}$ converges to f with respect to $\mathcal{T}((\mathcal{U}_{d_P})^s)$. However,

$$\sum_{n=0}^{\infty} 2^{-n} d^s(f(n), f_k(n)) = \sum_{n=k}^{\infty} 2^{-n} d^s(x, x_k) > 2$$

for all $k \in \mathbb{N}$. By Remark 2, $(f_k)_{k\in\mathbb{N}}$ does not cluster to f with respect to $\mathcal{T}((\mathcal{U}_{d_{\mathcal{B}_x}})^s)$, which provides a contradiction. Therefore, d is bounded.

Conversely, by hypothesis, there exists an $M > 0$ such that $d(y,z) \leq M$ for all $y, z \in X$. Now let $\varepsilon > 0$ and $U_\varepsilon = \{(f,g) \in \mathcal{B}_x \times \mathcal{B}_x \mid d_{\mathcal{B}_x}(f,g) < \varepsilon\}$. Choose an $m \in \omega$ such that $2^{-m}M < \varepsilon/2$. Then

$$V_\varepsilon = \{(f,g) \in \mathcal{B}_x \times \mathcal{B}_x \mid d(f(n),g(n)) < \varepsilon/4 \text{ for } n = 0,1,...,m\}$$

is an element of $\mathcal{U}_{d_P}$, and $V_\varepsilon \subseteq U_\varepsilon$because for every $(f,g) \in V_\varepsilon$, we have

$$\sum_{n=0}^{\infty} 2^{-n} d(f(n),g(n)) < \sum_{n=0}^{m} 2^{-n}\frac{\varepsilon}{4} + \sum_{n=m+1}^{\infty} 2^{-n}M < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.$$

Hence, $\mathcal{U}_{d_{\mathcal{B}_x}} \subseteq \mathcal{U}_{d_P}$ on $\mathcal{B}_x$ The conclusion now follows from Proposition 1.

Theorem 1. *Let (X,d) be a quasi-metric space and let $x \in X$. Then $(\mathcal{B}_x, d_{\mathcal{B}_x})$ is bicomplete if and only if (X,d) is bicomplete.*

Proof. Suppose that $(\mathcal{B}_x, d_{\mathcal{B}_x})$ is a bicomplete quasi-metric space and let $(x_k)_{k\in\mathbb{N}}$ be a Cauchy sequence in (X,d^s). For each $k \in \mathbb{N}$ define $f_k : \omega \to X$ such that $f_k(n) = x_k$ for all $n \in \omega$. Clearly, $f_k \in \mathcal{B}_x$ for all $k \in \mathbb{N}$. Given $\varepsilon > 0$ there is $k_\varepsilon \in \mathbb{N}$ such that $d(x_k, x_m) < \varepsilon/2$ for all $k, m \geq k_\varepsilon$. Hence, $d_{\mathcal{B}_x}(f_k, f_m) = 2d(x_k, x_m) < \varepsilon$ for all $k, m \geq k_\varepsilon$. Thus, the sequence $(f_k)_{k\in\mathbb{N}}$ converges to a function $f \in \mathcal{B}_x$ with respect to $\mathcal{T}((d_{\mathcal{B}_x})^s)$. Put $y = f(0)$. Then, $d^s(y, x_k) \to 0$ because for each $\varepsilon > 0$ there is an $m_\varepsilon \in \mathbb{N}$ such that $\sum_{n=0}^{\infty} 2^{-n} d^s(f(n), f_k(n)) < \varepsilon$ for all $k \geq m_\varepsilon$. Hence, $d^s(y, x_k) = d^s(f(0), f_k(0)) < \varepsilon$ for all $k \geq m_\varepsilon$.

Conversely, let $(f_k)_{k\in\omega}$ be a Cauchy sequence in $(\mathcal{B}_x, (d_{\mathcal{B}_x})^s)$. Consider the quasi-metric d_P on $\mathcal{B}_x$ defined as above. Then d_P induces the topology of pointwise convergence on $\mathcal{B}_x$. Since $d_P \leq d_{\mathcal{B}_x}$, $(f_k)_{k\in\omega}$ is a Cauchy sequence in the metric space $(\mathcal{B}_x, (d_P)^s)$. Then, for each $n \in \omega$, the sequence $(f_k(n))_{k\in\omega}$ is a Cauchy sequence in the complete metric space (X, d^s), so it is convergent to a point $x_n \in X$ with respect to the topology $\mathcal{T}(d^s)$.

Define a function $g : \omega \to X$, by $g(n) = x_n$ for all $n \in \omega$. We first prove that $g \in \mathcal{B}_x$:

Indeed, assume the contrary. Then, for each $j \in \mathbb{N}$ there is an $m_j \in \omega$ such that

$$(1) \qquad j < \sum_{n=0}^{m_j} 2^{-n} d^s(x, x_n).$$

On the other hand, since $(f_k)_{k\in\omega}$ is a Cauchy sequence in $(\mathcal{B}_x, (d_{\mathcal{B}_x})^s)$, there exists, by Remark 2, a $k_1 \in \omega$ such that for each $k \geq k_1$,

$$(2) \qquad \sum_{n=0}^{\infty} 2^{-n} d^s(f_k(n), f_{k_1}(n)) < 1.$$

Thus,

$$(2') \qquad \sum_{n=0}^{m_j} 2^{-n} d^s(f_k(n), f_{k_1}(n)) < 1$$

whenever $k \geq k_1$.

Let $j \in \mathbb{N}$. Then there exists a $k_0 \geq k_1$ such that $d^s(x_n, f_{k_0}(n)) < 2^{-j}$, for $n = 0, 1, ..., m_j$. Hence,

$$\text{(3)} \qquad \sum_{n=0}^{m_j} 2^{-n} d^s(x_n, f_{k_0}(n)) < 2^{-j} \sum_{n=0}^{m_j} 2^{-n} < 2^{-(j-1)}.$$

By (1), (3) and (2'), we obtain,

$$\begin{aligned} \text{(4)}j \quad &< \quad \sum_{n=0}^{m_j} 2^{-n} d^s(x, f_{k_0}(n)) + \sum_{n=0}^{m_j} 2^{-n} d^s(f_{k_0}(n), x_n) \\ &< \quad \sum_{n=0}^{m_j} 2^{-n} d^s(x, f_{k_0}(n)) + 2^{-(j-1)} < 2 + \sum_{n=0}^{m_j} 2^{-n} d^s(x, f_{k_1}(n)), \end{aligned}$$

which implies that

$$\sum_{n=0}^{\infty} 2^{-n} d^s(x, f_{k_1}(n)) = +\infty,$$

a contradiction. We conclude that $g \in \mathcal{B}_x$.

Finally, we prove that $(d_{\mathcal{B}_x})^s(g, f_k) \to 0$ as $k \to +\infty$. Indeed: Let $j \in \mathbb{N}$. Then there exists a $k(j) \in \omega$ such that for every $k, m \geq k(j)$,

$$\text{(5)} \qquad \sum_{n=0}^{\infty} 2^{-n} d^s(f_k(n), f_m(n)) < 2^{-(j+2)}.$$

Since both $f_{k(j)}$ and g are in $\mathcal{B}_x$, there exists $n_0 \in \omega$ (depending on j) such that

$$\begin{aligned} \text{(6)} \qquad\qquad n_0 - 1 \quad &> \quad j + 2, \\ \sum_{n=n_0}^{\infty} 2^{-n} d^s(x, f_{k(j)}(n)) \quad &< \quad 2^{-(j+2)}, \textit{and} \\ \sum_{n=n_0}^{\infty} 2^{-n} d^s(x, x_n) \quad &< \quad 2^{-(j+2)}. \end{aligned}$$

Since for each $n \in \omega$, $(f_k(n))_{k \in \omega}$ converges to x_n with respect to $\mathcal{T}(d^s)$, there exists a $k_j \geq k(j)$ such that for each $n \in \{0, 1, ..., n_0 - 1\}$ and each $k \geq k_j$, $d^s(x_n, f_k(n)) < 2^{-n_0}$. Therefore

$$\sum_{n=0}^{n_0-1} 2^{-n} d^s(x_n, f_k(n)) < 2^{-n_0} \sum_{n=0}^{n_0-1} 2^{-n} < 2^{-(n_0-1)} < 2^{-(j+2)}.$$

Moreover, it follows from (5) and (6), that for $k \geq k_j$,

$$\sum_{n=n_0}^{\infty} 2^{-n} d^s(x_n, f_k(n)) \leq \sum_{n=n_0}^{\infty} 2^{-n} d^s(x_n, x) + \sum_{n=n_0}^{\infty} 2^{-n} d^s(x, f_k(n))$$

$$
\begin{aligned}
&< \ 2^{-(j+2)} + \sum_{n=0}^{\infty} 2^{-n} d^s(x, f_{k(j)}(n)) + \sum_{n=0}^{\infty} 2^{-n} d^s(f_{k(j)}(n), f(_k(n)) \\
&< \ 2^{-(j+2)} + 2^{-(j+2)} + 2^{-(j+2)}.
\end{aligned}
$$

Thus we have shown that for each $j \in \mathbb{N}$ there is a $k_j \in \omega$ such that

$$\sum_{n=0}^{\infty} 2^{-n} d^s(g(n), f_k(n)) < 4 \cdot 2^{-(j+2)} \leq 2^{-j}$$

whenever $k \geq k_j$. We conclude that $(\mathcal{B}_x, d_{\mathcal{B}_x})$ is a bicomplete quasi-metric space.

Theorem 2. *Let (X,d) be a quasi-metric space and let $x \in X$. Then, $(\mathcal{B}_x, d_X)$ is bicomplete if and only if (X,d) is bicomplete.*

Proof. Suppose that (X, d) is bicomplete. Let $(f_k)_{k\in\mathbb{N}}$ be a Cauchy sequence in the extended metric space $(\mathcal{B}_x(d_X)^s)$. It follows from [5, Proposition 5] that $(f_k)_{k\in\mathbb{N}}$ converges to a function f in the extended metric space $(X^\omega, (d_X)^s)$. There is a $k_1 \in \mathbb{N}$ such that $(d_X)^s(f, f_{k_1}) < 1$. So, $\sum_{n=0}^{\infty} 2^{-n} d^s(x, f(n)) \leq \sum_{n=0}^{\infty} 2^{-n} d^s(x, f_{k_1}(n)) + \sum_{n=0}^{\infty} 2^{-n} d^s(f_{k_1}(n), f(n)) < +\infty$. Therefore, $f \in \mathcal{B}_x$. We conclude that $(\mathcal{B}_x, d_X)$ is an extended bicomplete quasi-metric space.

The converse follows similarly to the proof of the part 'only if' in Theorem 1.

Remark 4. In contrast to Theorem 2, there exists a bicomplete quasi-metric space (X, d) such that there is $x_0 \in X$ for which $(\mathcal{B}_{x_0}, d_P)$ is not bicomplete. Indeed, consider the bicomplete quasi-metric space $(\mathbb{R}^+, u)$ and let $x_0 = 0$. For each $k \in \mathbb{N}$ define the function $f_k : \omega \to \mathbb{R}^+$ given by $f_k(n) = 2^n - 1/2^k$ for all $n \in \omega$. It was proved in [9, Example 2] that $(f_k)_{k\in\mathbb{N}}$ is a nonconvergent Cauchy sequence in the metric space $(\mathcal{B}_0, (u_P)^s)$.

Proposition 4. *Let (X, d) be a weightable quasi-metric space which has a maximum x_0. Then, the function $W : \mathcal{C}_{x_0} \to \mathbb{R}^+$ defined by $W(f) = \sum_{n=0}^{\infty} 2^{-n} d(x_0, f(n))$, for all $f \in \mathcal{C}_{x_0}$, is a weighting function for the complexity space $(\mathcal{C}_{x_0}, d_{\mathcal{C}_{x_0}})$.*

Proof. We first note that if w is a weighting function for (X, d) then for each $n \in \omega$, we have

$$
\begin{aligned}
d(f(n), g(n)) + d(x_0, f(n)) &= d(f(n), g(n)) + w(f(n)) - w(x_0) \\
&= d(g(n), f(n)) + w(g(n)) - w(x_0) \\
&= d(g(n), f(n)) + d(x_0, g(n))
\end{aligned}
$$

because $d(x, x_0) = 0$ for all $x \in X$. Now let $f, g \in \mathcal{C}_{x_0}$. Then,

$$d_{\mathcal{C}_{x_0}}(f, g) + W(f) \ = \ \sum_{n=0}^{\infty} 2^{-n} d(f(n), g(n)) + \sum_{n=0}^{\infty} 2^{-n} d(x_0, f(n))$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} 2^{-n}[d(f(n), g(n)) + d(x_0, f(n))] \\
&= \sum_{n=0}^{\infty} 2^{-n}[d(g(n), f(n)) + d(x_0, g(n))] \\
&= \sum_{n=0}^{\infty} 2^{-n} d(g(n), f(n)) + \sum_{n=0}^{\infty} 2^{-n} d(x_0, g(n)) \\
&= d_{\mathcal{C}_{x_0}}(g, f) + W(g).
\end{aligned}
$$

We conclude that W is a weighting function for $(\mathcal{C}_{x_0}, d_{\mathcal{C}_{x_0}})$.

Theorem 3. *Let (X, d) be a weightable Smyth complete quasi-metric space which has a maximum x_0. Then, the complexity space $(\mathcal{C}_{x_0}, d_{\mathcal{C}_{x_0}})$ is Smyth complete.*

Proof. By Proposition 4, $(\mathcal{C}_{x_0}, d_{\mathcal{C}_{x_0}})$ is weightable, so it is Smyth completable. Since every bicomplete Smyth completable quasi-metric space is Smyth complete [4], the conclusion follows from Theorem 1.

Remark 5. The quasi-metric space $(\mathbb{R}^+, u)$ of the example in Remark 4 is Smyth complete and weightable and $x_0 = 0$ is the maximum for $(\mathbb{R}^+, u)$. Since for this space, $(\mathcal{C}_0, u_P)$ is not bicomplete, it follows that $(\mathcal{C}_0, \mathcal{U}_{u_P})$ is not Smyth complete. A similar situation occurs when one considers the quasi-uniformity of uniform convergence as the next example shows.

Example 6. Consider the weightable quasi-metric space $(\mathbb{R}^+, u)$. It is well known that $x_0 = 0$ is its maximum. Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{C}_0$, defined by $f_k(n) = 0$ if $k > n$ and $f_k(n) = 1$ otherwise. Then $u_X(f_k, f_{k+1}) = 0$ for all $k \in \mathbb{N}$. Thus, the filter generated by $\{\{f_k \mid k \geq m\} \mid m \in \mathbb{N}\}$ is a left K-Cauchy filter on $(\mathcal{C}_0, \mathcal{U}_{u_X})$. However, $u(f_{k+1}(k), f_k(k)) = 1$, so $u_X(f_{k+1}, f_k) = 1$ for all $k \in \mathbb{N}$. Thus $(f_k)_{k\in\mathbb{N}}$ has no limit in $\mathcal{T}(u^s)$. We conclude that $(\mathcal{C}_0, \mathcal{U}_{u_X})$ is not Smyth complete.

In the rest of this section we analyze the relationship between the quasi-uniformity of complexity convergence and the quasi-uniformity of compact convergence. In the light of Proposition 3 and the well known fact that the quasi-uniformity of uniform convergence coincides with the quasi-uniformity of compact convergence whenever the domain space is compact, this relationship is easily obtained in many interesting cases.

Let us recall that if $(Z, \mathcal{T})$ is a topological space and $(X, \mathcal{U})$ is a quasi-uniform space, the quasi-uniformity of compact convergence is the quasi-uniformity $\mathcal{U}_K$ on X^Z which has as a subbase the family the sets of the form

$$(K, U) = \{(f, g) \in X^Z \times X^Z \mid (f(x), g(x)) \in U forallx \in K\}.$$

It is well known that $\mathcal{U}_P \subseteq \mathcal{U}_K \subseteq \mathcal{U}_X$.

Let (X, d) be a weightable quasi-metric space with a maximum x_0 and let $\mathcal{T}$ be a compact topology on ω. Then, the quasi-uniformity of compact convergence $\mathcal{U}_{d_K}$

coincides with the quasi-uniformity of uniform convergence $\mathcal{U}_{d_X}$, so by Proposition 2, $\mathcal{U}_{d_K}$ coincides with the quasi-uniformity of complexity convergence $\mathcal{U}_{d_{\mathcal{C}_{x_0}}}$ only if $X = \{x_0\}$.

In the case that $\mathcal{T}$ is the discrete topology on ω, the quasi-uniformity of compact convergence coincides with the quasi-uniformity of pointwise convergence, so, by Proposition 3, $\mathcal{U}_{d_K}$ coincides with $\mathcal{U}_{d_{\mathcal{C}_{x_0}}}$ if and only if d is bounded.

3 The Grothendieck theorem for the complexity space $(\mathcal{C}_0, u_0)$

For an arbitrary Tychonoff topological space X we denote, as usual, by $C_p(X)$ the space of all continuous real-valued functions on X with the topology of pointwise convergence.

The celebrated Grothendieck theorem states that if X is a Tychonoff countably compact space and A is a subset of $C_p(X)$ such that every infinite subset of A has a limit point in $C_p(X)$, then the closure of A is compact in $C_p(X)$ [3]. Asanov and Velichko [1] have obtained the following interesting generalization of Grothendieck's theorem: if X is a Tychonoff countably compact space, then the closure in $C_p(X)$ of every bounded subset A of $C_p(X)$ is compact.

In Theorem 4 below, we shall obtain a version of Asanov-Velichko's theorem in our context.

We first recall that a subset A of a topological space $(X, \mathcal{T})$ is said to be bounded if every continuous real-valued function on X is bounded on A.

Similarly, we define:

Definition 2. Let $(X, \mathcal{T})$ be a topological space. A subset A of X is called *upper bounded* if every upper semicontinuous real-valued function on X is upper bounded on A.

It is well known that a subset A of a metrizable space X is bounded if and only if every sequence in A has a cluster point in X. The following result is an easy consequence of this fact.

Proposition 5. *Let $\mathcal{F}$ be a bounded subset of the (metrizable) space $(\mathcal{C}_0, \mathcal{T}(u_{\mathcal{C}_0})^s)$. Then the closure of $\mathcal{F}$ in $(\mathcal{C}_0, (u_{\mathcal{C}_0})^s)$ is compact in $(\mathcal{C}_0, (u_{\mathcal{C}_0})^s)$.*

In the next we will focus our attention on the interesting case where $\mathcal{F}$ is an upper bounded subset of the (quasi-metrizable) space $(\mathcal{C}_0, \mathcal{T}(u_{\mathcal{C}_0}))$ and we will prove the somewhat surprising result (compare Remark 6 below) that every upper bounded subset of $(\mathcal{C}_0, \mathcal{T}(u_{\mathcal{C}_0}))$ is bounded in $(\mathcal{C}_0, \mathcal{T}((u_{\mathcal{C}_0})^s))$, so the conclusion obtained in Proposition 5 also will hold in the upper bounded case. Actually, we will state a more general result (see Theorem 4, below).

According to our notation we denote by $((\mathbb{R}^+)^\omega, (u_P)^s)$ the metric space consisting of all functions from ω to $\mathbb{R}^+$ endowed with the metric $(u_P)^s$, where by u_P we denote the quasi-metric (of pointwise convergence) defined on $(\mathbb{R}^+)^\omega \times (\mathbb{R}^+)^\omega$

by $u_P(f,g) = \sum_{n=0}^{\infty} 2^{-n} \min\{1, u(f(n), g(n))\}$.

We will need two lemmas.

Lemma 1. *A subset A of a quasi-metrizable space X is upper bounded if and only if every sequence in A has a cluster point in X.*

Proof. Suppose that A is an upper bounded subset of the quasi-metrizable space X such that there is a sequence $(a_n)_{n\in\omega}$ in A without cluster point (in X). Then $\cap_{n=0}^{\infty} F_n = \emptyset$, where for each $n \in \omega$, F_n denotes the closure in X of the set $\{a_k \mid k \geq n\}$. For each $n \in \omega$ let f_n be the characteristic function of $X \backslash F_n$. Then each f_n is lower semicontinuous in X, and thus $f = \sum_{n=0}^{\infty} 2^{-n} f_n$ is also a lower semicontinuous function in X. Clearly, $f(x) > 0$ for all $x \in X$ and $f(a_n) \leq 2^{-n}$ for all $n \in \omega$. Hence, $g = 1/f$ is an upper semicontinuous function in X which is not upper bounded on A, a contradiction. Therefore, $(a_n)_{n\in\omega}$ clusters to some point in X.

Conversely, suppose that A is not upper bounded in the quasi-metrizable space X. Then there is an upper semicontinuous function f on X and a sequence $(a_n)_{n\in\omega}$ in A such that $f(a_n) > n$ for all $n \in \omega$. It follows that $(a_n)_{n\in\omega}$ has no cluster point in X. The proof is complete.

Remark 6. It is easy to show that if (X, d) is a quasi-metric, then upper boundedness in $(X, \mathcal{T}(d))$ does not imply boundedness in $(X, \mathcal{T}(d^s))$, in general. For instance, it follows from Lemma 1 that $(-\infty, 0]$ is an upper bounded subset of $(\mathbb{R}, \mathcal{T}(u))$; however, it is clear that it is not bounded in $(\mathbb{R}, \mathcal{T}(u^s))$.

Lemma 2. *Let $(f_k)_{k\in\omega}$ be a sequence in $\mathcal{C}_0$ that converges to a function $f \in \mathcal{C}_0$ with respect to the topology $\mathcal{T}(u_{\mathcal{C}_0})$. Then $(f_k)_{k\in\omega}$ clusters to a function $g \in \mathcal{C}_0$ with respect to the topology $\mathcal{T}((u_{\mathcal{C}_0})^s)$. Moreover, $g(n) \leq f(n)$ for all $n \in \omega$.*

Proof. Since $(f_k)_{k\in\omega}$ converges to f with respect to $\mathcal{T}(u_{\mathcal{C}_0})$, there is $k_1 \in \omega$ such that for each $k \geq k_1$, $\sum_{n=0}^{\infty} 2^{-n} u(f(n), f_k(n)) < 1$. So $\sum_{n=0}^{\infty} 2^{-n} f_k(n) < 1 + \sum_{n=0}^{\infty} 2^{-n} f(n)$ for all $k \geq k_1$. Thus $f_k(n) < 2^n M$ for all $n \in \omega$ and for all $k \geq k_1$, where $M = 1 + \sum_{n=0}^{\infty} 2^{-n} f(n) < +\infty$. Therefore, the subset $\{f_k \mid k \geq k_1\}$ of $\mathcal{F}$ is contained in the compact space $\Pi_{n=0}^{\infty} [0, 2^n M]$, and, hence, there is a subsequence $(g_k)_{k\in\omega}$ of $(f_k)_{k\in\omega}$ and a $g \in (\mathbb{R}^+)^{\omega}$ such that $(g_k)_{k\in\omega}$ converges to g with respect to the topology of pointwise convergence $\mathcal{T}((u_P)^s)$.

Next we note that $g \in \mathcal{C}_0$. Indeed, since $u_{\mathcal{C}_0}(f, g_k) \to 0$, we have $u_P(f, g_k) \to 0$, so, by the triangle inequality, $u_P(f, g) = 0$, i.e. $g(n) \leq f(n)$ for all $n \in \omega$. Therefore $g \in \mathcal{C}_0$ because $f \in \mathcal{C}_0$.

It remains to show that $(g_k)_{k\in\omega}$ converges to g with respect to $\mathcal{T}((u_{\mathcal{C}_0})^s)$. Let $j \in \mathbb{N}$. Then there is a $k(j) \in \omega$ such that

$$\sum_{n=0}^{\infty} 2^{-n} u(f(n), g_k(n)) < 2^{-(j+2)} \text{ for all } k \geq k(j). \tag{7}$$

Moreover, since both f and g are in $\mathcal{C}_0$, there is $n_0 \in \omega$, with $n_0 - 1 > j + 2$, such that $\sum_{n=n_0}^{\infty} 2^{-n} f(n) < 2^{-(j+2)}$ and $\sum_{n=n_0}^{\infty} 2^{-n} g(n) < 2^{-(j+2)}$.

Therefore, by the convergence of $(g_k)_{k\in\omega}$ to g with respect to $\mathcal{T}((u_P)^s)$, there is a $k_j \geq k(j)$ such that for each $n \in \{0, 1, ..., n_0-1\}$ and each $k \geq k_j$, $\mid g(n)-g_k(n) \mid < 2^{-n_0}$.

Consequently,

$$\sum_{n=0}^{n_0-1} 2^{-n} \mid g(n) - g_k(n) \mid < 2^{-n_0} \sum_{n=0}^{n_0-1} 2^{-n} < 2^{-(n_0-1)} < 2^{-(j+2)}$$

for all $k \geq k_j$.

On the other hand, by (7), we have $\sum_{n=n_0}^{\infty} 2^{-n}u(f(n), g_k(n)) < 2^{-(j+2)}$ for all $k \geq k_j$, and from this inequality it easily follows that $\sum_{n=n_0}^{\infty} 2^{-n}g_k(n) < 2^{-(j+2)} + \sum_{n=n_0}^{\infty} 2^{-n}f(n)$, and thus $\sum_{n=n_0}^{\infty} 2^{-n}g_k(n) < 2 \cdot 2^{-(j+2)}$ for all $k \geq k_j$. Hence,

$$\sum_{n=0}^{\infty} 2^{-n} \mid g(n) - g_k(n) \mid < 2^{-(j+2)} + \sum_{n=n_0}^{\infty} 2^{-n}g(n) + \sum_{n=n_0}^{\infty} 2^{-n}g_k(n) <$$

$4 \cdot 2^{-(j+2)} \leq 2^{-j}$ for all $k \geq k_j$.

We conclude, by Remark 2, that $(g_k)_{k\in\omega}$ converges to g with respect to $\mathcal{T}((u_{\mathcal{C}_0})^s)$.

Corollary 1. *Let $\mathcal{F}$ be an upper bounded subset of $(\mathcal{C}_0, \mathcal{T}(u_{\mathcal{C}_0}))$. Then $\mathcal{F}$ is a bounded subset of $(\mathcal{C}_0, \mathcal{T}((u_{\mathcal{C}_0})^s))$.*

Proof. By Lemmas 1 and 2, every sequence in $\mathcal{F}$ clusters to a function in $\mathcal{C}_0$ with respect to the topology $\mathcal{T}((u_{\mathcal{C}_0})^s)$. Therefore, $\mathcal{F}$ is bounded in $(\mathcal{C}_0, \mathcal{T}((u_{\mathcal{C}_0})^s))$.

By Corollary 1 and Proposition 5 we deduce the following version of Grothendieck-Asanov-Velichko's theorem.

Let $\mathcal{F}$ be an upper bounded subset of $(\mathcal{C}_0, \mathcal{T}(u_{\mathcal{C}_0}))$. Then the closure of $\mathcal{F}$ in $(\mathcal{C}_0, (u_{\mathcal{C}_0})^s)$ is compact in $(\mathcal{C}_0, (u_{\mathcal{C}_0})^s)$.

However, it is possible to obtain a more general result as our final theorem shows.

Theorem 4. *Let $\mathcal{F}$ be an upper bounded subset of $(\mathcal{C}_0, \mathcal{T}(u_{\mathcal{C}_0}))$. Then the closure of $\mathcal{F}$ in $(\mathcal{C}_0, (u_{\mathcal{C}_0})^{-1})$ is compact in $(\mathcal{C}_0, (u_{\mathcal{C}_0})^s)$.*

Proof. Denote by $\overline{\mathcal{F}}$ the closure of $\mathcal{F}$ in the quasi-metric space $(\mathcal{C}_0, (u_{\mathcal{C}_0})^{-1})$. Let $(h_k)_{k\in\omega}$ be a sequence in $\overline{\mathcal{F}}$. Then there is a sequence $(f_k)_{k\in\omega}$ in $\mathcal{F}$ such that $u_{\mathcal{C}_0}(f_k, h_k) < 2^{-k}$ for all $k \in \omega$. By Corollary 1, there exist a subsequence $(f_{k_m})_{m\in\omega}$ and an $f \in \mathcal{C}_0$ such that $(u_{\mathcal{C}_0})^s(f, f_{k_m}) \to 0$, so $u_{\mathcal{C}_0}(f, h_{k_m}) \to 0$. By Lemma 2, $(h_{k_m})_{m\in\omega}$ clusters to a function $g \in \mathcal{C}_0$ with respect to $\mathcal{T}((u_{\mathcal{C}_0})^s)$. Therefore g clusters to $(f_k)_{k\in\omega}$ with respect to $\mathcal{T}((u_{\mathcal{C}_0})^{-1})$, so $g \in \overline{\mathcal{F}}$. This completes the proof.

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Escuela de Caminos, Departamento de Matemática Aplicada, Universidad Politécnica de Valencia, Apartado 22012, 46071 Valencia, Spain.
E-mail: sromague@mat.upv.es

Department of Computation, National University of Ireland, Cork, Ireland.
E-mail: m.schellekens@cs.ucc.ie