Domains are quanti able

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OVERVIEW

Origins of Quantitative Domain Theory

Main tools: partial metrics and valuations

Correspondence between partial metrics and valuations

Domains are quanti able

Conclusion & future work

ORIGINS OF QUANTITATIVE DOMAIN THEORY

Scott's Domain Theory is based on complete **partial orders**.

An alternative approach at CWI uses **metric spaces**. The attraction of this approach is that the link with Topology is made more explicit.

Order approach = **qualitative**"
Metric approach = **quantitative**"

To reconcile the order theoretic and the metric approach, Mike Smyth used generalized metrics, known as **quasi-metrics**". Quasi-metrics nicely incorporate the notion of a partial order and establish a direct link with topology.

QUASI-METRICS

Quasi-metrics are obtained from classical metrics by **removing the symmetry requirement**. In general:

$$d(x; y) \neq d(y; x).$$

Quasi-metrics nicely reconcile the order and metric approaches to Domain Theory since they have an associated partial order:

$$x \sqsubseteq_d y \iff d(x; y) = 0.$$

Topological completions based on quasi-metrics generalize some standard domain theory completions such as the "cpo-completion" and the "ideal completion". (Smyth, Sunderhauf, Flagg, Kopperman and Rutten).

Remark: general quasi-metrics are *hard* to handle, since the loss of symmetry leads to a great variety of alternative approaches (e.g. Cauchy sequences).

QUASI-METRIC SPACES

$d : X \times X \to R_0^+$ is a *quasi-metric* iff

1) $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$.
2) $d(x, y) + d(y, z) \geq d(x, z)$

Examples:

1) $d_1 : R^2 \to R_0^+$, defined by: $d_1(x, y) = y - x$ when $x < y$; $d_1(x, y) = 0$ otherwise.

2) (CWI approach) A partial order $(X, \sqsubseteq)$ can be *encoded* via a quasi-metric space $(X, d_{\sqsubseteq})$, defined via a 0-1 valued distance $d_{\sqsubseteq}$:

$$d_{\sqsubseteq}(x, y) = 0 \Leftrightarrow x \sqsubseteq y.$$

QUASI-METRICS AND DOMAIN THEORY

Each quasi-metric space $(X;d)$ induces:

1) a generalized **Alexandro topology** T_d

generated by the base $\{B_{\epsilon}[x] \mid \epsilon > 0; x \in X\}$, where $B_{\epsilon}[x] = \{y \mid d(x;y) < \epsilon\}$

2) a **partial order** $\sqsubseteq_d$

$$x \sqsubseteq_d y \Leftrightarrow d(x;y) = 0$$

3) an **associated metric** d^*

$$d^*(x;y) = max\{d(x;y); d(y;x)\}$$

In practice we wish $(X;\sqsubseteq_d)$ to be a domain and T_d to be the Scott topology.

FOUNDATIONS BASED ON QUASI-METRICS

Historicaly three foundations for Quantitative Domain Theory are available:

Generalized Metric Spaces, Yoneda completion (Bonsangue, Rutten, Van Breugel)

Continuity Spaces (Flagg, Kopperman)

Topological Quasi-Uniform Spaces (Smyth, Sunderhauf)

Simpli cation:
Totally Bounded (quasi-metric) Spaces (Smyth)

Paper available: \On the Yoneda completion of a quasi-metric space", Kunzi, Schellekens, http://www.cs.ucc.ie/ mpcs/, TCS.

PROBLEMS WITH THE FOUNDATIONS

In order to carry out standard topological completions, continuity spaces and topological quasi-uniform spaces need to be equipped with a second topology. This leads to serious technical complications. On the other hand, the Yoneda completion leads to non idempotent completions.

The totally bounded quasi-metric spaces introduced by Mike Smyth allow for standard topological completions. These spaces incorporate important classes of domains, but *not* all domains. I.e. not every domain allows for a totally bounded quasi-metric which induces the Scott topology.

THE QUANTIFICATION PROBLEM

"Is there a generalized metric which allows for standard topological completions *and* which generates the Scott topology?".

The totally bounded spaces have a "weak symmetry" property, which explains their elegant behavior regarding completions.

We will see that a special class of quasi-metric spaces, the partial metric spaces, solves the quanti cation problem. These spaces are more general than the totally bounded spaces and also have the weak symmetry property.

QUANTITATIVE MEASURES

Recent developments in Domain theory use additional concepts in order to develop the applications.

The applications include domain theoretic models for data ow networks, logic programming, complexity, integration, real number computation and probabilistic languages.

Each of these applications involve \quantitive measures" in some sense.

The question remained how \quantitive measures" could be introduced to classical Domain Theory in an elegant and uniform way.

APPLICATIONS

Data ow Networks (Matthews 1994)
PARTIAL METRIC SPACES

Logic Programming (Seda 1996)
TOTALLY BOUNDED SPACES

Domain theoretic treatment integration
VALUATIONS (Edalat 1995)
PARTIAL METRIC SPACES (Heckman)

Models for e ciency analysis
PARTIAL METRIC SPACES
(Schellekens 1995)

Models for real number computation
PARTIAL METRIC SPACES
(Escardo 1997)

COMPARISON OF DIFFERENT APPROACHES

The applications listed above, indicate that the following concepts play an important role:

Partial metric spaces

Totally bounded spaces

Valuations

Once can show that totally bounded spaces are partial metrizable. Recent work, to which we return in the coming slides, has shown that partial metrics and valuations are tightly related.

PARTIAL METRICS

\Partial metric $p(x;y)$ = generalized metric for which distance between a point and itself need not be zero."

EQUIVALENT FORMULATION IN TERMS OF QUASI-METRICS

A quasi-metric space $(X;d)$ is **weightable** i $9w: X \; ! \; R_0^+:$

$$d(x;y) + w(x) = d(y;x) + w(y):$$

Example: $(R_0^+;d_1;w_1)$ where w_1 is the identity function on R_0^+.

A quasi-metric space $(X;d)$ is **co-weightable** i $9w: X \; ! \; R_0^+:$

$$d(y;x) + w(x) = d(x;y) + w(y):$$

THE COMPLEXITY SPACE

The *complexity space* $(\mathcal{C}, d_{\mathcal{C}})$ allows one to use Semantics techniques (fixed point analysis) to analyse the complexity of algorithms.

$$\mathcal{C} = \{f : \omega \to \overline{\mathbb{R}}^{+} \mid \sum_{n=0}^{\infty} 2^{-n} \frac{1}{f(n)} < +\infty\}$$

and $d_{\mathcal{C}}$ is the quasi-metric on $\mathcal{C}$ defined by

$$d_{\mathcal{C}}(f, g) = \sum_{n=0}^{\infty} 2^{-n} \left[\left(\frac{1}{g(n)} - \frac{1}{f(n)}\right) \vee 0\right]$$

whenever $f, g \in \mathcal{C}$. The complexity space $(\mathcal{C}, d_{\mathcal{C}})$ is a quasi-metric space with a maximum $\top$, which is the function with constant value ∞. The complexity space $(\mathcal{C}, d_{\mathcal{C}})$ is weightable by the function $w_{\mathcal{C}}$ where $\forall f \in \mathcal{C} : w_{\mathcal{C}}(f) = \sum_n \frac{2^{-n}}{f(n)}$.

THE DOMAIN OF STREAMS

^{1} are the, possibly nite, sequences of elements from a given set . The partial metric p is de ned as follows: $8x;y\ 2\ ^{1}:$

$$p(x;y) = 2 \quad ;$$

where $= maxfnj\,x(n) = y(n)g$ when the sequences x and y have a common non empty initial subsequence and $= 0$ otherwise.

The function $p(x;y) \quad p(x;x)$ is the \Baire quasi-metric", weightable by $w(x) = p(x;x)$.

THE INTERVAL DOMAIN

The interval domain $(\mathbf{I}(R), \mathbf{p})$ consists of the closed intervals of the reals ordered by reverse inclusion and equipped with the partial metric $\mathbf{p}$ defined by:

$$\mathbf{p}([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}.$$

The function $\mathbf{p}([a,b],[c,d]) - \mathbf{p}([a,b],[a,b])$ is a quasi-metric, weightable by $\mathbf{w}([a,b]) = \mathbf{p}([a,b],[a,b])$.

DOMAINS ARE QUANTIFIABLE

De nition 1 *A domain is partial metrizable i there exists a partial metric which induces the Scott topology. A domain is strongly partial-metrizable i it is partial-metrizable and its associated metric induces the Lawson topology.*

Theorem 2 *!-continuous dcpo's are partial metrizable. The class of strongly partial metrizable domains coincides with the 2/3 SFP domains.*

OVERVIEW PROOF

1) Establish a one to one correspondence between partial metrics and generalized valuations.

2) Provide a characterization of partial metrizability for a special class of semilattices in terms of the existence of a generalized valuation.

3) Finally, obtain the partial metrizability of domains by showing that the lattice of Scott closed sets of each domain is partial metrizable. The partial metric on the domain is inherited from the partial metric on this lattice.

TOPOLOGICAL PROBLEM

Open problem in Non Symmetric Topology:

Kunzi: \Characterize those quasi-uniformities having a countable base which are induced by a weighted quasi-pseudo-metric (= a partial metric)" (Problem 7, survey paper: Non symmetric topology)

Main stumbling block: \partial metric spaces do not seem to embody as yet enough of the structure of the examples arising in the applications."

Solution: isolate a \mathematically nice" subclass of partial metric spaces, which is still su ciently large to incorporate the domain theoretic examples.

COMPUTER SCIENCE PROBLEM

Birkho : Metrics can be generated from valuations.

O'Neil: Partial metrics can be generated from valuations.

Bukatin: \The existence of deep connections between partial metrics and valuations is known in Domain Theory".

SEMIVALUATIONS ON SEMILATTICES

VALUATIONS

SEMIVALUATIONS

CHARACTERIZATION OF VALUATIONS IN TERMS OF SEMIVALUATIONS

MOTIVATION

Valuations are useful for modelling non deterministic computations and real number computations

Probabilistic Powerdomain
(Jones, Plotkin 89)

Lower bag domain (Heckmann 94)

Domain theoretic treatment of integration (Edalat 94)

Birkho : \... many of the most important applications of lattices to mathematics involve limiting processes like those of real analysis. Such processes can be de ned in many ways, ... The simplest way is in terms of \valuations", ... ".

VALUATIONS

A *join (meet) semilattice* is a partial order $(X, \leq)$ such that every two elements $x, y \in X$ have a supremum $x \sqcup y$ (infimum $x \sqcap y$) in X.

Let L be a lattice.
A function $f : L \to \mathbb{R}^+$ is a *valuation* iff

(1) f is increasing.

(2) $\forall x, y \in L$: $f(x \sqcap y) + f(x \sqcup y) = f(x) + f(y)$.

Examples: 1) Cardinality function on a powerset lattice $(P(A), \subseteq)$. 2) Probability measures.

Main objective: to generalize the fruitful notion of a valuation on a lattice to the context of semilattices.

Non trivial problem: semilattices only provide a single operation!

SEMIVALUATIONS

If $(X;\le)$ is a join semilattice then a function $f:(X;\le)\to R_0^+$ is *join valuation* iff

$$\forall x;y;z \in X: f(x\sqcup z) \le f(x\sqcup y)+f(y\sqcup z) - f(y):$$

Comment: Join valuations are increasing.

Indeed: let z = x and assume $x \le y$. Then $f(x) \le f(y) + f(y) - f(y) = f(y)$ and thus $f(x) \le f(y):$

If $(X;\le)$ is a meet semilattice then a function $f:(X;\le)\to R_0^+$ is a *meet valuation* iff

$$\forall x;y;z \in X: f(x\sqcap z) \ge f(x\sqcap y)+f(y\sqcap z) - f(y):$$

Comment: Meet valuations are increasing.

SEMIVALUATION SPACES

A function is a *semivaluation* if it is either a join valuation or a meet valuation. A join (meet) valuation space is a join (meet) semilattice equipped with a join (meet) valuation. A *semivaluation space* is a semilattice equipped with a semivaluation.

Example: Let $(X; \le)$ be a meet semilattice, A a countable subset of X and let $w\colon X \to R_0^+$ be a function such that

$$\sum_{a \in A} w(a) < \infty .$$

The function $\overline{w}$ is defined by:

$$\forall x \in X \colon \overline{w}(x) = \sum_{a \in A} \{ w(a) \mid a \in x\!\downarrow \},$$

where $x\!\downarrow = \{ a \in A \mid a \le x \}$, is a meet valuation on $(X; \le)$.

SEMI-MODULARITY

Proposition 3 *Let L be a lattice.*

1) A function $f : L \to R_0^+$ is a join valuation if and only if it is increasing and satisfies join-modularity, i.e.:

$$f(x \sqcup z) + f(x \sqcap z) \quad f(x) + f(z):$$

2) A function $f : L \to R_0^+$ is a meet valuation if and only if it is increasing and satisfies meet-modularity, i.e.

$$f(x \sqcup z) + f(x \sqcap z) \quad f(x) + f(z):$$

CHARACTERIZATION

Corollary 4 *A function on a lattice is a valuation i it is a join valuation and a meet valuation.*

This result clearly establishes that semivaluations provide suitable generalizations of valuations from the context of lattices to the context of semilattices!

CORRESPONDENCE BETWEEN PARTIAL METRICS AND SEMIVALUATIONS

QUASI-METRIC SEMILATTICES

CORRESPONDENCE

QUASI-METRIC SEMILATTICES

A quasi-metric space is a semilattice iff the associated order is a semilattice.

The terminology of *quasi-metric (quasi-uniform) semilattice* is reserved for quasi-pseudo-metric (quasi-uniform) spaces which are semilattices for which the operations are quasi-uniformly continuous.

A join semilattice $(X; d)$ is *invariant* iff $\forall x; y; z \in X : d(x \sqcup z; y \sqcup z) \leq d(x; y)$.

Invariant join semilattices are quasi-metric join semilattices and similar definitions apply for the case of invariant meet semilattices and for invariant lattices.

EXAMPLES

I), II) Scott domains, represented as totally bounded spaces or as 0-1 valued quasi-metric spaces.

III) Baire partial metric spaces

IV) Complexity spaces

V) Any quasi-metric space with an associated linear order

Conclusion: The quasi-metric semilattices (more generally: the quasi-uniform semilattices) provide a mathematical nice class, su ciently large to incorporate many relevant examples.

Of course domains in general are NOT semilattices.

CORRESPONDENCE I

Theorem 5 *For every join semilattice* $(X; \quad)$, *there exists a bijection between invariant weighted quasi-metrics* d *on* X *with* $\quad_d = \quad$ *and fading strictly decreasing join co-valuations* $f : (X; \quad) \to (R_0^+; \quad)$.

The map $f \mapsto d_f$ *is defined by*

$$d_f(x;y) = f(y) \quad f(x \sqcup y):$$

The inverse is the function which to each weighed space $(X; d)$ *associates its unique fading weighting.*

CORRESPONDENCE II

Theorem 6 *For every meet semilattice* $(X;$), *there exists a bijection between invariant co-weighted quasi-metrics* d *on* X *with* $_d =$ *and fading strictly increasing meet valuations* $f : (X; \ ) \ ! \ (R_0^+; \ )$.

The map f 7! d_f *is defined by*

$$d_f(x;y) = d_f(x;y) = f(x) \quad f(x \, u \, y):$$

The inverse is the function which to each weighed space $(X;d)$ *associates its unique fading co-weighting.*

Remark: We motivate the choices for the terminology \join valuation" and \join co-valuation". We chose to reserve the terminology \join co-valuation" for weightings rather than for co-weightings since co-weightings are increasing and hence are in accordance with the traditional computer science convention which defines valuations as increasing functions.

CORRESPONDENCE III

Corollary 7 *For every lattice* (***X*;**), *there exists a bijection between invariant weighted quasi-metrics* ***d*** *on* ***X*** *with* $_{d} =$ *and fading strictly decreasing modular functions on the lattice and there exists a bijection between invariant co-weighted quasi-metrics* ***d*** *on* ***X*** *with* $_{d} = v$ *and fading valuations on the lattice.*

A CHARACTERIZATION OF PARTIAL METRIZATION

SEMI-CO-VALUATIONS

ORDER QUASI-UNIMORPHISMS

Q-VALUATIONS

SOLUTION PROBLEM 7

SEMI-CO-VALUATIONS

De nition 8 *If* $(X; \quad)$ *is a join semilattice then a function* $f : (X; \quad) \; ! \; R_0^+$ *is a join-valuation i*

$$f(x \; t \; z) \qquad f(x \; t \; y) + f(y \; t \; z) \qquad f(y)$$

and f *is a join-co-valuation i*

$$f(x \; t \; z) \qquad f(x \; t \; y) + f(y \; t \; z) \qquad f(y):$$

If $f : (X; \quad) \; ! \; R_0^+$ *is a function on a meet semilattice* $(X; \quad)$ *then* f *is a meet-valuation i*

$$f(x \; u \; z) \qquad f(x \; u \; y) + f(y \; u \; z) \qquad f(y)$$

and f *is a meet-co-valuation i*

$$f(x \; u \; z) \qquad f(x \; u \; y) + f(y \; u \; z) \qquad f(y):$$

The terminology \positive" and \negative" indicate \strictly increasing" and \strictly decreasing" respectively.

ORDER QUASI-UNIMORPHISMS

A **quasi-uniform space** is a pair $(\boldsymbol{X}; U)$ consisting of a set $\boldsymbol{X}$ with a lter U such that

1) $8\boldsymbol{U}\ 2\ U\colon \quad \boldsymbol{U}$
2) $8\boldsymbol{U}\ 2\ U\ 9\boldsymbol{V}\ 2\ U\colon \boldsymbol{V} \quad \boldsymbol{V} \quad \boldsymbol{U}$.

If $(\boldsymbol{X}; U)$ and $(\boldsymbol{Y}; V)$ are quasi-uniform spaces then a function $\boldsymbol{f}\colon(\boldsymbol{X}; U)\ !\ (\boldsymbol{Y}; V)$ is an **order quasi-unimorphism** i

1) $\boldsymbol{f}$ is surjective
2) $\boldsymbol{f}$ is strictly increasing
3) $8\boldsymbol{V}\ 9\boldsymbol{U}\colon \boldsymbol{x} \quad _U\, \boldsymbol{y}\ \ )\ (\boldsymbol{xUy}\ \ )\ (\boldsymbol{f}\boldsymbol{x})\boldsymbol{V}(\boldsymbol{f}\boldsymbol{y}))$
4) $8\boldsymbol{V}\ 9\boldsymbol{U}\colon \boldsymbol{x} \quad _U\, \boldsymbol{y}\ \ )\ (\boldsymbol{f}(\boldsymbol{x})\boldsymbol{U}\boldsymbol{f}(\boldsymbol{y})\ \ )\ \boldsymbol{xVy})$.

Order quasi-unimorphisms with range space $(R_0^+; U_{\boldsymbol{d}_1})$ and $(R_0^+; U_{\boldsymbol{d}_1}{}^1)$ are respectively referred to as

left & right order quasi-unimorphisms.

Q-VALUATIONS

A **Q-meet valuation** on a quasi-uniform meet semilattice is a meet valuation which is a right order quasi-unimorphism.

A **Q-meet co-valuation** is a meet co-valuation which is a left order quasi-unimorphism.

Solution to Problem 7 of Kunzi's Survey paper Non Symmetric Topology:

Theorem 9 *If* $(\boldsymbol{X}; U)$ *is a quasi-uniform join semilattice for which U has a countable base then U is generated by a weightable invariant quasi-metric , there exists a Q-join co-valuation on* $(\boldsymbol{X}; U)$.

Theorem 10 *If* $(\boldsymbol{X}; U)$ *is a quasi-uniform meet semilattice for which U has a countable base then U is generated by a co-weightable u-invariant quasi-metric , there exists a Q-meet valuation on* $(\boldsymbol{X}; U)$.

DOMAINS ARE QUASI-METRIZABLE

Definition 11 *For a domain* $(X, \sqsubseteq)$ *with a basis* $B = (a_n)_n$*, the Smyth-quasi-metric* d_S *is defined by:*

$$d_S(x, y) = \inf\{\frac{1}{2^n} \mid \forall i \le n : a_i \ll x \Rightarrow a_i \ll y\}.$$

Proposition 12 *Domains* $(P, \sqsubseteq)$ *are quasi-metrizable by the quasi-metric* d_S*; i.e.* d_S *induces the Scott topology on* $(P, \sqsubseteq)$*.*

For the case of ω-algebraic domains, the quasi-metric d_S can be simplified by replacing the way below inequality “$\ll$” by “$\sqsubseteq$”.

Proposition 13 *2/3 SFP* ω*-algebraic domains* (P, d_S)*, equipped with the Smyth quasi-metric* d_S*, are totally bounded.*

THE LATTICE OF SCOTT CLOSED SETS

Definition 14 *Let $C(\boldsymbol{P})$ denote the lattice of Scott-closed subsets of $\boldsymbol{P}$, ordered by inclusion.*

We remark that $\boldsymbol{P}{\downarrow} = \{\boldsymbol{x}{\downarrow} \mid \boldsymbol{x} \in \boldsymbol{P}\} \subseteq C(\boldsymbol{P})$.

Definition 15 *The Smyth quasi-metric is extended to the lattice of Scott-closed sets $C(\boldsymbol{P})$ by:*

$$\boldsymbol{D}_{\boldsymbol{S}}^{\boldsymbol{c}}(\boldsymbol{C}, \boldsymbol{C}') = \inf_{\boldsymbol{n}} \{ \frac{1}{2^{\boldsymbol{n}}} \mid \forall \boldsymbol{i} \leq \boldsymbol{n}.\ \boldsymbol{a_i} \in \boldsymbol{C} \Rightarrow \boldsymbol{a_i} \in \boldsymbol{C}' \}.$$

The restriction $\boldsymbol{d}_{\boldsymbol{S}}^{\boldsymbol{c}}$ of $\boldsymbol{D}_{\boldsymbol{S}}^{\boldsymbol{c}}$ to $\boldsymbol{P}$ is defined by:
$\boldsymbol{d}_{\boldsymbol{S}}^{\boldsymbol{c}}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{D}_{\boldsymbol{S}}^{\boldsymbol{c}}(\boldsymbol{x}{\downarrow}, \boldsymbol{y}{\downarrow})$.

THE EXTENDED QUASI-METRIC

We leave the straightforward veri cations of the following three lemmas to the reader.

Lemma 16 D_S^c *is a join-invariant quasi-metric and hence* $(C(P); U_{D_S^c})$ *is a quasi-uniform join semilattice, equipped with the subset order, i.e.* $C \quad_{D_S^c} C' \quad , \quad C \quad C'$.

For the following Lemma we use the fact that $(x\#)+ = x+$.

Lemma 17 d_S^c *coincides with the Smyth quasi-metric* d_S.

Lemma 18 *Let* $(P; v)$ *be a domain with countable base* $B = (a_n)_n$. *We use the following notation for* $A \quad P$: $8n: A[n] = A \setminus f a_i j\, i \quad n g$. *Then, for any two Scott-closed sets,* C *and* C' *such that* $C \quad C'$*, we have:*

$$D_S^c(C; C') = \inf_n f \frac{1}{2^n} j\, (C+)[n] = (C'+)[n] g:$$

DOMAINS ARE QUANTIFIABLE

In the following proposition, on the quantifiability of domains $(P; \sqsubseteq)$ with a countable base $B = \{a_n\}_n$, we use the notation d^w to distinguish this quasi-metric from the quasi-metric d_w defined in Theorem 5. In fact, d^w is d_{K-w} (using the notation of Theorem 5), where $K = \sum_{a_n} w(a_n)$.

Proposition 19 *Let $(P; \sqsubseteq)$ be a domain with a countable base $B = (a_n)_n$ and let $w: B \to \mathbb{R}^+$ denote a function such that $\forall n: w(a_n) > 0$ and $\sum_{a_n \in B} w(a_n) < \infty$. Then $(P; \sqsubseteq)$ is quantifiable by the following bi-weightable quasi-metric:*

$$d^w(x; y) = \sum_{a_n \in x^{+} \setminus y^{+}} w(a_n).$$

In case the domain has a least element $\perp$, one can allow $w(\perp) = 0$.

OMEGA-ALGEBRAIC CASE

For the case of omega-algebraic domains, the proposition can be simpli ed to:

If $(P; v)$ is $!$-algebraic, then $(P; v)$ is quanti able by the following bi-weightable quasi-metric:

$$d^w(x; y) = \sum_{a_n 2\, x\# \quad y\#} w(a_n):$$

In this case, the associated metric d induces the Lawson topology.

QUANTIFICATION OF A DOMAIN

Definition 20 *Given a domain* $(P, \sqsubseteq)$ *with countable base* B. *A function* $w: B \to R^+$ *which has finite sum over* B *is called a basic valuation. The valuation*

$$W(C) = \sum_{a_n \in C} w(a_n)$$

is called the valuation generated by w *and the weighted quasi-metric*

$$d^w(x, y) = \sum_{a_n \in x^+ - y^+} w(a_n)$$

is called the quasi-metric generated by w. *The corresponding partial metric*

$$p^w(x, y) = \sum_{a_n \notin x^+ \cap y^+} w(a_n)$$

is called the partial metric generated by w.

For any domain $(P, \sqsubseteq)$, *we refer to the partial metric space* (P, p^w) *as a quantification of the domain.*

QUANTIFICATIONS

We obtain the following immediate corollary of Proposition 27.

Corollary 21 *Quanti cations* $(\boldsymbol{P};\boldsymbol{p}^{w_1})$ *and* $(\boldsymbol{P};\boldsymbol{p}^{w_2})$ *of a domain* $(\boldsymbol{P};v)$ *are equivalent, i.e. the quasi-metrics* $\boldsymbol{d}^{w_1}$ *and* $\boldsymbol{d}^{w_2}$ *generate the same quasi-uniformity.*

Corollary 22 *For* **!***-algebraic dcpo's* $\boldsymbol{P}$ *the following holds:* $\boldsymbol{P}$ *is 2/3 SFP i any of its quanti cations* $(\boldsymbol{P};\boldsymbol{p}^{w})$ *is compact.*