The ideal completion is not sequentially adequate

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Abstract

It is well known that for the case of a countable partial order, the ideal completion and the chain completion coincide. We investigate the boundary at which the chain and ideal completion do not coincide. We show in particular that the ideal completion is not sequentially adequate; that is it is not possible in general to simply replace the ideal completion with a completion based on sequences as for instance the chain completion. The implications of this result for the Yoneda completion ([BvBR98]) and for the Smyth completion ([Smy89],[Smy91],[Smy94],[Sün93] and [Sün95]) which are based on the ideal completion, are discussed in an extended version of this paper, reported in [KS98].

The authors acknowledge the support by the Swiss National Science Foundation, 21-30585.91, 20-50579.97 and 2000-041475.94/2 respectively, the last of which has funded a research stay of the second author at the University of Berne.

1 The ideal completion

We recall the definition of the ideal completion (e.g. [DP90]) and of the sequential version of the ideal completion known as the chain completion.

If $(P, \sqsubseteq)$ is a partial order and A is a nonempty subset of P, then A is an *ideal* iff $\forall y \in A.\, x \sqsubseteq y \Rightarrow x \in A$ and A is directed; that is $\forall x, y \in A\, \exists z \in A.\, x \sqsubseteq z$ and $y \sqsubseteq z$.

The *ideal completion* of a partial order $(P, \sqsubseteq, \bot)$ with a least element $\bot$, is the partial order $(Q, \subseteq, \{\bot\})$ where Q is the set of all ideals.

Let $(P, \leq)$ be a partial order. A sequence $(x_n)_n$ in P is *eventually increasing* iff $\exists n_0\, \forall m, n \geq n_0.\, m \leq n \Rightarrow x_m \leq x_n$. We let $S(P)$ denote the set of eventually increasing sequences for this partial order.

We remark that in the following we will use the standard terminology "chain completion" as used in theoretical computer science (e.g. [BvBR98]), where the notion of a chain refers to a countable linear order. This replaces the standard mathematical definition of a chain as a linear order.

The *chain completion* of a partial order $(P, \leq)$ is defined to be the partial order $(\overline{P}^i, \overline{\sqsubseteq}^i)$, where $\overline{P}^i = S(P)_{/\approx^i}$ and where:

$\forall (x_n)_n \in S(P)\ \forall (y_m)_m \in S(P).$

$$(x_n)_n \sqsubseteq^i (y_m)_m \Leftrightarrow \exists n \, \forall k \geq n \, \forall m \, \exists l \geq m.\, x_k \leq y_l$$

$$(x_n)_n \approx^i (y_m)_m \Leftrightarrow (x_n)_n \sqsubseteq^i (y_m)_m \text{ and } (y_m)_m \sqsubseteq^i (x_n)_n$$

$\forall [(x_n)_n] \in \overline{P}^i\ \forall [(y_m)_m] \in \overline{P}^i.$

$$[(x_n)_n] \ \overline{\sqsubseteq}^i \ [(y_m)_m] \Leftrightarrow (x_n)_n \sqsubseteq^i (y_m)_m.$$

Remarks:
1) It is easy to verify that the relation $\approx$ is an equivalence relation.

2) It is well known (e.g. [BvBR98]) that for countable partial orders, the ideal completion and the chain completion coincide.

An equivalent version of the chain completion, frequently encountered in the literature (e.g. [CV74]) is the following:

The chain completion of a partial order $(P, \leq)$ is defined to be the pair $(\overline{P}, \overline{\sqsubseteq})$, where $\overline{P} = S(P)_{/\approx}$ and where:

$\forall (x_n)_n \in S(P)\ \forall (y_m)_m \in S(P).$

$$(x_n)_n \sqsubseteq (y_m)_m \Leftrightarrow \exists k' \, \forall k \geq k' \, \exists l' \, \forall l \geq l'.\, x_k \leq y_l$$

$$(x_n)_n \approx (y_m)_m \Leftrightarrow (x_n)_n \sqsubseteq (y_m)_m \text{ and } (y_m)_m \sqsubseteq (x_n)_n$$

$\forall [(x_n)_n] \in \overline{P}\ \forall [(y_m)_m] \in \overline{P}.$

$$[(x_n)_n] \ \overline{\sqsubseteq} \ [(y_m)_m] \Leftrightarrow (x_n)_n \sqsubseteq (y_m)_m.$$

2 Sequential inadequacy

We recall the following well-known result (e.g. [BvBR98]) for which the verification is straightforward.

Lemma 1 *For a countable partial order, the chain completion and the ideal completion coincide.*

We will show in the following that the ideal completion of a partial order, in general is not replaceable by a sequential completion; in other words, sequences are not adequate for the ideal completion. Hence the ideal completion is in general not replaceable by the chain completion.

We recall the definition of an (ω-)algebraic partially ordered set (e.g. [Mis91] and [Smy91]).

Definition 2 *An element e of a partially ordered set $(P, \sqsubseteq)$ is finite if for each directed subset D of P for which $\sqcup D$ exists, $e \leq \sqcup D$ implies that $e \leq d$ for some $d \in D$. The set of finite elements of P is denoted by $F(P)$ and for each $x \in P$, $F(x) = \{e \in F(P) | \, e \sqsubseteq x\}$.*

An algebraic partially ordered set is a partially ordered set $(P, \sqsubseteq)$ satisfying the property that for each $x \in P$, $F(x)$ is directed and that $x = \sqcup F(x)$.

An ω-algebraic partially ordered set $(P, \sqsubseteq)$ is an algebraic partially ordered set such that $F(P)$ is at most countable.

We recall some of the basic theory of ordinals (e.g. [HJ84]). A set A is *transitive* iff $\forall x, y \in A. \, x \in y \in A \Rightarrow x \in A$. A set A is is *well ordered by the membership relation* iff $(A, \in)$ is a total order and every nonempty subset of A has a least element. A set α is an *ordinal* iff α is transitive and well ordered by the membership relation. The *successor* of an ordinal α is the ordinal $\alpha + 1$ defined by $\alpha + 1 = \alpha \cup \{\alpha\}$. An ordinal α is called a *successor ordinal* iff $\alpha = \beta + 1$ for some ordinal β. Otherwise α is called a *limit ordinal*. Limit ordinals can be characterized as the ordinals α such that $\cup \alpha = \alpha$.

We denote the first uncountable ordinal by ω_1. The empty set $\emptyset$ is an ordinal denoted by 0.

We recall that the partial ordering on an ordinal α is the subset-order $\subseteq$ and that each ordinal consists of the ordinals which strictly precede it in the subset-order, where we have that $\alpha \subset \beta \Leftrightarrow \alpha \in \beta$.

Since every ordinal α is a total order, it is clear that every subset A of α is directed. Also, every subset A of α possesses a supremum which is $\cup A$.

Lemma 3 *Every ordinal is algebraic and the finite elements of a non zero ordinal α are given by the set $\{\gamma + 1 | \, \gamma + 1 \in \alpha\} \cup \{0\}$.*

Proof: The fact that 0 is algebraic follows by a straightforward verification.

For a given non zero ordinal α, we verify first that the set $\{\gamma + 1 | \, \gamma + 1 \in \alpha\} \cup \{0\}$ consists of finite elements.

The fact that 0 is finite is trivial since each ordinal contains the ordinal 0 as a subset.

Let $\beta \in \alpha$ be a successor ordinal, where say $\beta = \gamma + 1$. We show that β is finite. Assume that D is a (directed) subset of α and that $\beta = \gamma + 1 \subseteq \cup D$, where say $\cup D = \nu$. We obtain that $\gamma \subset \cup D$, i.e. $\gamma \in \cup D$, and thus $\gamma \in \mu$ for some $\mu \in D$.

We distinguish two cases, depending on whether ν is a successor or a limit.

For the first case, we assume that ν is a successor ordinal, say $\rho + 1$. Then one can easily verify that $\nu \in D$. Indeed, otherwise if $\nu \notin D$, we obtain that $\forall \alpha \in D. \alpha \in \nu = \rho + 1$ and thus $\forall \alpha \in D. \alpha \subseteq \rho$ which implies that $\cup D \subseteq \rho$. Hence we have the contradiction that $\rho + 1 \subseteq \rho$. Hence $\nu \in D$.

We recall that $\gamma \in \mu \subseteq \nu = \rho + 1$. In particular we have that $\gamma \in \nu$ and thus $\gamma \subseteq \rho$. Hence $\beta = \gamma + 1 \subseteq \rho + 1 = \nu$, which implies that $\beta \subseteq \nu$, where $\nu \in D$. Hence $\gamma + 1$ is finite.

For the second case, we assume that ν is a limit ordinal. We recall that $\beta = \gamma + 1 \subseteq \cup D = \nu$. Since ν is by assumption a limit ordinal, we obtain that $\gamma + 1 \neq \nu$ and thus $\gamma + 1 \in \nu = \cup D$. Hence $\gamma + 1 \in \mu$ for some $\mu \in D$ and thus $\gamma + 1$ is finite.

To show the converse, we need to verify that every finite element of α is a successor ordinal or 0. Since we have remarked that 0 is a finite element, it suffices to show that every finite element e of α which is not 0, is a successor ordinal. We assume by way of contradiction that e is a finite element which is a non zero limit ordinal. In that case we obtain that $e = \cup e$. It is straightforward to verify that this fact, combined with the fact that $e \neq 0$, implies that e is not finite.

Finally, we need to show that every ordinal α is algebraic. For this we need to show that each element β of α is the supremum of a subset of the set $\{\gamma+1 | \gamma+1 \in \alpha\} \cup \{0\}$. Let β be an element of α. In case β is a successor ordinal, the result follows trivially. So we can assume that β is a limit ordinal. In that case we have that $\beta = \cup \beta$. Clearly if $\beta = 0$ then $\beta = \cup 0$ and thus we can assume that $\beta \neq 0$.

Since every limit ordinal $\mu \in \beta$ is such that $\mu + 1 \in \beta$, we obtain that $\beta = \cup\{\nu+1 | \nu \in \beta\}$. In other words, each limit ordinal β is the supremum of the successor ordinals below β and thus β is the supremum of the set of finite elements below β.

Hence we have shown that each ordinal is algebraic.

□

Lemma 4 *ω_1 is the first ordinal which is not ω-algebraic.*

Proof: To verify that any ordinal α strictly smaller than ω_1 is ω-algebraic, it suffices, by Lemma 3, to verify that the set of the finite elements $F(\alpha)$ is countable. This last fact however follows since any ordinal α strictly below ω_1 is countable.

Next, we verify that the ordinal ω_1 is not ω-algebraic. Indeed, by Lemma 3, the set of finite elements $F(\omega_1)$ is the set of the successor ordinals of ω_1 supplemented by the element 0. Since ω_1 is a limit ordinal, we obtain for each of its elements α which is a limit ordinal, that $\alpha + 1$ belongs to ω_1. Hence the set of limit ordinals of ω_1 has a cardinality below the cardinality of the set of successor ordinals of ω_1. So if ω_1 would have countably many finite elements then ω_1 would be countable. Hence ω_1 is not ω-algebraic.

□

Definition 5 *An algebraic partial order is sequentially adequate iff every element of this partial order is the supremum of an eventually increasing sequence of finite elements.*

Lemma 6 *Every ω-algebraic partial order is sequentially adequate.*

Proof: We present a sketch. The argument is similar to the one of [DP90], exercise 3.5 of Chapter 3.

Given an ω-algebraic partial order $(P, \sqsubseteq)$. Let x be any element of P and let D be a directed subset of finite elements of P such that $x = \sqcup D$. Since $F(P)$ is countable, we obtain that D is countable, say $D = \{x_0, x_1, \ldots, x_n, \ldots\}$. For each finite subset F of D, let u_D be an upper bound of F.

Inductively define subsets D_i of F as follows:

$$D_0 = \{x_0\} \text{ and } D_{i+1} = D_i \cup \{y_{i+1}, u_{D_i \cup \{y_{i+1}\}}\},$$

where y_{i+1} is the element x_n in $D - D_i$ with subscript n chosen as small as possible. One can verify that the sequence $(u_i)_i$, defined by $\forall i \geq 0.\, u_i = u_{D_i \cup \{y_{i+1}\}}$, is an increasing sequence in D with supremum $\sqcup D$.

□

The following lemma implies that the converse of Lemma 6 does not hold, since we obtain that the ordinal ω_1 is a sequentially adequate algebraic partial order which is not ω-algebraic.

Lemma 7 $\omega_1 + 1$ *is the first ordinal which is not sequentially adequate.*

Proof: To show that $\omega_1 + 1$ is not sequentially adequate, we argue by contradiction.

If $\omega_1 + 1$ were sequentially adequate, then in particular its maximum ω_1 would be the supremum of a sequence of finite elements from $\omega_1 + 1$. Without loss of generality we can assume that this sequence does not contain the ordinal 0. Hence, by Lemma 9, the sequence consists of successor ordinals. Since ω_1 is a limit ordinal, it does not belong to this sequence and hence all elements from the sequence are strictly below ω_1. Since ω_1 is the first countable ordinal, each of these finite elements is countable. However, since the union of countably many countable sets is countable, we obtain the contradiction that ω_1 is countable. Hence $\omega_1 + 1$ is not ω-algebraic.

The verification that every ordinal strictly below $\omega_1 + 1$ is sequentially adequate proceeds in two steps.

First we remark that, by Lemma 4, any ordinal strictly below ω_1 is ω-algebraic and hence sequentially adequate.

Next we verify that ω_1 is sequentially adequate. Consider any element $\beta \in \omega_1$. We remark that β is countable. If β is a successor ordinal, it follows that β is a finite element and hence trivially is the limit of a countable sequence of finite elements. In case β is a limit ordinal, we obtain, via an argument similar to the end of the proof of Lemma 3, that β is the limit of the successor ordinals below β. Since β is countable, this implies that β is the limit of a countable sequence of finite elements. Hence ω_1 is sequentially adequate.

□

Before stating the conclusion, we need a final lemma on the ideal completion of an ordinal.

Lemma 8 *The ideal completion of an ordinal α is its successor ordinal $\alpha + 1$.*

Proof: The ideals of an ordinal α are its downwardly closed directed subsets. It is easy to verify that these are precisely the ordinals less than or equal to α. Hence it is easy to see that the ideal completion of α is $\alpha + 1$.

□

Corollary 9 *The ideal completion is not sequentially adequate.*

Proof: We present a sketch. Consider the ordinal ω_1 which has $\omega_1 + 1$ as its ideal completion. We remark that $\omega_1 + 1$ is not sequentially adequate by Lemma 7. From the proof of Lemma 7, we know that there is no sequence of finite elements which converges to ω_1; that is there is no sequence of successor ordinals which converges to ω_1. Hence, by an argument similar to the one at the end of the proof of Lemma 3, there is no sequence of ordinals strictly below ω_1 and converging to ω_1.

So we have shown that there is no sequential completion which can replace the general ideal completion.

□

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