A polynomial time algorithm for a class of Quantified Integer Programs

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Abstract. It is well known that the Quantified Satisfiability problem (QSAT) is PSPACE-complete. It follows that the problem of deciding the language of 0/1 Quantified Integer Programs (QIPs) i.e., testing whether a linear system of inequalities has a quantified lattice point is PSPACE-complete. One aspect of research is to focus on designing polynomial time procedures for interesting special cases. In this paper, we show that if the constraint matrix defining a 0/1 QIP is totally unimodular (TUM), then the QIP can be decided in polynomial time .

1 Introduction

Quantified decision problems are useful in modeling situations, wherein a policy (action) can depend upon the effect of imposed stimuli. A typical such situation is a $2-$ person game. Consider a board game comprised of an initial configuration and two players A and B each having a finite set of moves. A can win the game if the decision problem: *Given the initial configuration, does A have a first move (policy), such that for all possible first moves of B (imposed stimulus), A has a second move, such that for all possible second moves of B,... , A eventually wins?* can be answered affirmatively. The board configuration can be represented as as a boolean expression or a constraint matrix; the effort involved in representing the board configuration typically determines the tractability of the decision problem.

Definition 1. *Let $\{x_1, x_2, \ldots, x_n\}$ be a set of n boolean variables. A disjunction of literals (a literal is either x_i or its complement $\bar{x_i}$) is called a* clause*, represented by C_i. A satisfiability problem of the form:*

$$Q_1 x_1 Q_2 x_2 \ldots Q_n x_n \; C \tag{1}$$

where each Q_i is either a $\exists$ or $\forall$ and $C = C_1 \wedge C_2 \ldots \wedge C_m$, is called a Quantified Satisfiability (QSAT) problem.

QSAT has been shown to be PSPACE-complete, even when there are at most 3 literals per clause (Q3SAT) [Pap94], although polynomial time algorithms exist for the case in which there are at most two literals per clause [APT79,Gav93].

Definition 2. *Let $x_1, x_2, \ldots x_n$ be a set of n 0/1 variables. An integer program of the form*

$$Q_1 x_1 \in \{0,1\} Q_2 x_2 \in \{0,1\}, \ldots Q_n x_n \in \{0,1\} \mathbf{A}.\vec{\mathbf{x}} \leq \vec{\mathbf{b}}? \tag{2}$$

where each Q_i is either $\exists$ or $\forall$ is called a 0/1 Quantified Integer Program (QIP).

The PSPACE-completeness of QIPs follows directly from the PSPACE-completeness of QSAT; in fact the reduction from QSAT to QIP is identical to the one from SAT to 0/1 Integer Programming. The matrix $\mathbf{A}$ is called the *constraint matrix* of the QIP. Without loss of generality, we assume that the quantifiers are strictly alternating, $Q_1 = \exists$; further we denote the existentially quantified variables using $x_i, i = 1, 2, \ldots, n$ and the universally quantified variables using $y_i, i = 1, 2, \ldots, n$. Thus we can write an arbitrary 0/1 QIP as :

$$\exists x_1 \in \{0,1\} \forall y_1 \in \{0,1\} \exists x_2 \in \{0,1\} \forall y_2 \in \{0,1\} \ldots \exists x_n \in \{0,1\} \forall y_n \in \{0,1\} \mathbf{A}.[\vec{\mathbf{x}} \;\; \vec{\mathbf{y}}]^{\mathbf{T}} \leq \vec{\mathbf{b}}? \tag{3}$$

for suitably chosen $\vec{\mathbf{x}}, \vec{\mathbf{y}}, \mathbf{A}, \vec{\mathbf{b}}, n$

Definition 3. *A TQIP is a QIP in which the constraint matrix is totally unimodular.*

Definition 4. *A linear program of the form*

$$\exists x_1 \in [0,1] \forall y_1 \in [0,1] \exists x_2 \in [0,1] \forall y_2 \in [0,1] \ldots \exists x_n \in [0,1] \forall y_n \in [0,1] \mathbf{A}.[\vec{\mathbf{x}} \ \ \vec{\mathbf{y}}]^{\mathbf{T}} \leq \vec{\mathbf{b}}? \tag{4}$$

is called a 0/1 *Quantified Linear Program (QLP).*

Definition 5. *A TQLP is a QLP in which the constraint matrix is totally unimodular.*

The complexity of QLPs (0/1 or otherwise) is not known [Joh], although the class of TQLPs can be decided in polynomial time [Sub01a] (See §A).

2 Algorithms and Complexity

Lemma 1.

$$\begin{aligned} & \mathbf{L} : \exists x_1 \in \{0,1\} \forall y_1 \in \{0,1\} \ldots \exists x_n \in \{0,1\} \forall y_n \in \{0,1\} \mathbf{A}.[\vec{\mathbf{x}} \ \ \vec{\mathbf{y}}]^{\mathbf{T}} \leq \vec{\mathbf{b}} \\ \Leftrightarrow \ \ & \mathbf{R} : \exists x_1 \in \{0,1\} \forall y_1 \in [0,1] \ldots \exists x_n \in [0,1] \forall y_n \in [0,1] \mathbf{A}.[\vec{\mathbf{x}} \ \ \vec{\mathbf{y}}]^{\mathbf{T}} \leq \vec{\mathbf{b}} \end{aligned} \tag{5}$$

Proof: $\mathbf{R} \Rightarrow \mathbf{L}$ *is trivial. We focus on* $\mathbf{L} \Rightarrow \mathbf{R}$. *Pick some vector* $\vec{\mathbf{y'}} \in \{0,1\}^n$*; let* $\vec{\mathbf{x'}} = [x'_1, x'_2, \ldots, x'_n]^T = [c_0, f_1(y'_1), f_2(y'_1, y'_2), \ldots, f_{n-1}(y'_1, y'_2, \ldots, y'_{n-1})]$ *be such that* $\mathbf{A}.[\vec{\mathbf{x'}} \ \ \vec{\mathbf{y'}}]^{\mathbf{T}} \leq \vec{\mathbf{b}}$ *(where the* f_i *are the Skolem functions capturing the dependence of* x_i *on* $y'_1, y'_2, \ldots, y'_{i-1}$ *and* c_0 *is a constant in* $[0,1]$*). Likewise, pick a second vector* $\vec{\mathbf{y''}} \in \{0,1\}^n$ *and let* $\vec{\mathbf{x''}} = [x''_1, x''_2, \ldots, x''_n]^T = f_{n-1}(y''_1, y''_2, \ldots, y''_{n-1})]$*, such that* $\mathbf{A}.[\vec{\mathbf{x''}} \ \ \vec{\mathbf{y''}}]^{\mathbf{T}} \leq \vec{\mathbf{b}}$*. Now consider the parametric point*
$\vec{\mathbf{y'''}} = \lambda.\vec{\mathbf{y'}} + (\mathbf{1} - \lambda).\vec{\mathbf{y''}}, 0 \leq \lambda \leq 1$. *We shall show that the parametric point* $\vec{\mathbf{x'''}} = \lambda.\vec{\mathbf{x'}} + (\mathbf{1} - \lambda).\vec{\mathbf{x''}}, 0 \leq \lambda \leq 1$ *is such that* $\mathbf{A}.[\vec{\mathbf{x'''}} \ \ \vec{\mathbf{y'''}}]^{\mathbf{T}} \leq \vec{\mathbf{b}}$. *Observe that* $\mathbf{A}.[\vec{\mathbf{x'''}} \ \ \vec{\mathbf{y'''}}]^{\mathbf{T}} = \mathbf{A}.[\lambda.\vec{\mathbf{x'}} + (\mathbf{1} - \lambda).\vec{\mathbf{x''}} \ \ \lambda.\vec{\mathbf{y'}} + (\mathbf{1} - \lambda).\vec{\mathbf{y''}}]^{\mathbf{T}} = \mathbf{A}.[\lambda.\vec{\mathbf{x'}} \ \ \lambda.\vec{\mathbf{y'}}]^{\mathbf{T}} + \mathbf{A}.[(\mathbf{1} - \lambda).\vec{\mathbf{x''}} \ \ (\mathbf{1} - \lambda).\vec{\mathbf{y''}}]^{\mathbf{T}} = \lambda.\mathbf{A}[\vec{\mathbf{x'}} \ \ \vec{\mathbf{y'}}]^{\mathbf{T}} + (\mathbf{1} - \lambda).\mathbf{A}[\vec{\mathbf{x''}} \ \ \vec{\mathbf{y''}}]^{\mathbf{T}} \leq \lambda.\vec{\mathbf{b}} + (\mathbf{1} - \lambda).\vec{\mathbf{b}} \leq \vec{\mathbf{b}}$, *since* $0 \leq \lambda \leq 1$. *Thus the feasible solution space of a Quantified Linear Program is convex and the lemma is proven.* □

Lemma 2.

$$\begin{aligned} & \mathbf{L} : \exists x_1 \in \{0,1\} \forall y_1 \in \{0,1\} \ldots \exists x_n \in \{0,1\} \forall y_n \in \{0,1\} \mathbf{A}.[\vec{\mathbf{x}} \ \ \vec{\mathbf{y}}]^{\mathbf{T}} \leq \vec{\mathbf{b}} \\ \Leftrightarrow \ \ & \mathbf{R} : \exists x_1 \in [0,1] \forall y_1 \in \{0,1\} \ldots \exists x_n \in [0,1] \forall y_n \in \{0,1\} \mathbf{A}.[\vec{\mathbf{x}} \ \ \vec{\mathbf{y}}]^{\mathbf{T}} \leq \vec{\mathbf{b}} \end{aligned} \tag{6}$$

Proof: *Consider any vector* $\vec{\mathbf{y}} = \{0,1\}^n$. *Substituting this vector in System (3) results in a standard integer program of the form* $\exists \vec{\mathbf{x}} = \{0,1\}^n \, \mathbf{G}.\vec{\mathbf{x}} \leq \vec{\mathbf{d}}$, *where* $\mathbf{G}$ *is totally unimodular. Consequently. this system has a solution if and only if the system* $\exists \vec{\mathbf{x}} = [0,1]^n \, \mathbf{G}.\vec{\mathbf{x}} \leq \vec{\mathbf{d}}$ *is feasible and Lemma (2) follows.* □

Theorem 1. *TQIPs can be relaxed to TQLPs, while preserving the integrality of the solution space and hence can be decided in polynomial time.*

Proof: *Use Lemma (1) to relax the universally quantified variables and Lemma (2) to relax the existentially quantified variables to get a TQLP; then use Algorithm (A.1) in Appendix §A to decide the TQLP in polynomial time.* □

3 Conclusion

The technique used in this paper is different from the one used in [Sub01b] to provide a polyhedral projection procedure to decide Quantified 2−SAT problems.

A Deciding Quantified Linear Programs

In this section, we outline the strategy used in [Sub01a] to solve QLPs. The principal idea underlying Algorithm (A.1) is the elimination of the quantified variables while preserving the solution space. Elimination of a universally quantified variable leaves the number of constraints unchanged, whereas the elimination of an existentially quantified variable using a strategy such as Fourier-Motzkin elimination could lead to a quadratic increase in the number of constraints (see [Sch87]); consequently Algorithm (A.1) could take exponential time in the worst case. In the case of TQLPs though, it runs in time $O(n^5 . \log n)$, where n represents the number of variables in the QLP.

Fast convergence in TQLPs is guaranteed by the following lemma

Lemma 3. *Given a totally unimodular matrix* $\mathbf{A}$ *of dimensions* $m \times n$*, for a fixed* n*,* $m = O(n^2)$*, if each row is unique.*

Proof: *The above lemma was proved for a superset of totally unimodular matrices viz. totally balanced matrices in [Ans80,AF84]. It therefore follows that Lemma (3) is true.* □

The import of Lemma (3) is that a totally unimodular constraint matrix cannot have more than $O(n^2)$ non-redundant constraints. The elimination of an existentially quantified variable through Fourier-Motzkin elimination could potentially result in $O(n^4)$ constraints. Eliminating the redundant constraints is a sort operation, that can be implemented in time $O(n^5 . \log n)$ time [1].

Function QLP-DECIDE $(\mathbf{A}, \vec{\mathbf{b}}, \mathbf{Q})$

```
{The array Q stores the quantifiers i.e. Q[i] = Q_i}
for ( i = n down to 1 ) do
  if ( Q[i] = ∃ ) then
    ELIM-UNIV-VARIABLE(y_i)
    if (CHECK-INCONSISTENCY()) then
      return ( false )
    end if
    PRUNE-CONSTRAINTS()
  else
    ELIM-EXIST-VARIABLE(x_i)
    if (CHECK-INCONSISTENCY()) then
      return ( false )
    end if
  end if
end for
System is feasible
return
```

Algorithm A.1: A Quantifier Elimination Algorithm for deciding Query **E**

Function ELIM-UNIV-VARIABLE $(\mathbf{A}, \vec{\mathbf{b}}, i)$

```
Substitute x_i = 0 in each constraint that can be written in the form x_i ≥ ()
Substitute x_i = 1 in each constraint that can be written in the form x_i ≤ ()
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Algorithm A.2: Eliminating Universally Quantified variable $x_i \in [0, 1]$

The procedure ELIM-EXIST-VARIABLE is implemented through the polyhedral projection algorithm known as the Fourier-Motzkin elimination procedure [Sch87] as discussed above.

[1] $O(n^4)$ row vectors can be sorted in time $n^4 . \log n^4$; each comparison takes $O(n)$ time.

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