Interpretation of Propositional Logic

An interpretation for PL is a function from the propositional letters to {T, F}

Interpretation I **satisfies** a formula A if the formula evaluates to T

- Written as: $\models_I A$

A is **valid** (tautology) if *every* interpretation satisfies A

- Written as: $\models A$

A set S is **satisfiable** if *some* interpretation satisfies every formula in S

Satisfiability

- A set S of statements is satisfiable (or consistent) if *some* interpretation satisfies *all elements of S* at the same time.

example of a consistent set

$$\{X > 1,\ Y > 5,\ X+Y < 12\}$$

- Otherwise S is unsatisfiable (or inconsistent)

example of an inconsistent set:

$$\{A < B,\ B < C,\ A > C\}$$

Using the Semantics

Semantic Entailment or Logical Consequence

- A set S of statements entails a statement A

  if every interpretation that satisfies all elements of S also satisfies A.
- Written as $S \models A$

Example:

- $\{A < B,\ B < C\} \models A < C$

$S \models A$ if and only if $\{\neg A\} \cup S$ is inconsistent

$\models A$ if and only if A is valid

Using the Rules

Inference

Want to check A is valid

- Can do this by checking all interpretations
- But what if there are infinitely many?

Let $\{A_1, \ldots. A_n\} \models B$.

If $A_1, \ldots. A_n$ are true then B must be true.

Write this as the inference rule:

$$\frac{A_1, \ldots. A_n}{B}$$

Use inferences to construct finite proofs

Semantic Entailment(Logical Consequence)

- A set S of statements entails a statement A if every interpretation that satisfies all elements of S also satisfies A.
- Written as S |= A

|= A if and only if A is valid

Deducibility(Proof by Inference)

An inference rule: $\dfrac{H_1, \ldots H_n}{C}$

Use inferences to construct finite proofs

- A is deducible from the set S if there is a finite proof of A starting from elements of S
- Written as: S |- A

Soundness and Completeness

Soundness theorem: If S $\vdash$ A then S $\models$ A

Completeness theorem: If S $\models$ A then S $\vdash$ A

Proof: another picture

Axiom1

Axiom2

Axiom3

Rule of Inference

Rule of Inference

theoremA

theoremB

theoremC

Schematic Inference Rules

Rules are general, using schema variables:

$$\frac{X \text{ part of } Y \qquad Y \text{ part of } Z}{X \text{ part of } Z}$$

A valid inference:

$$\frac{\text{spoke part of wheel} \qquad \text{wheel part of bike}}{\text{spoke part of bike}}$$

An inference may be valid even if the premises are false:

$$\frac{\text{cow part of cheese} \qquad \text{cheese part of moon}}{\text{cow part of moon}}$$