Implication

…a closer look at this important connective

Intuitive meaning:

- If I haven't finished the first project report then I will study during the break.

$$\text{NonFinishRep} \rightarrow \text{StudyDurBreak}$$

- Explain situations (i),(ii),(iii), (iv)?

Implication

- The *converse* of $p \rightarrow q$ is $q \rightarrow p$
  – The converse is a **different** unrelated statement
- The *contrapositive* of $p \rightarrow q$ is $\neg q \rightarrow \neg p$
  – The contrapositive does hold (is equivalent)

$$\neg StudyDurBreak \rightarrow \neg NonFinishRep$$

Truth Tables

- Check a formula by constructing a truth table to consider all possible values for the propositional variables
- tautology
  - formula is always true
- contradiction
  - formula is always false

Truth table

- ….

X	Y		$X \wedge Y$	$\neg X$	$\neg Y$	$(\neg X \vee \neg Y) \leftrightarrow \neg(X \wedge Y)$
F	F		F	T	T	
F	T		F	T	F	
T	F		F	F	T	
T	T		T	F	F	

Logical Equivalence

- Two formulae are logically equivalent if they have the same truth table
- Truth table shows that for all possible values the formulae(statements) have the same truth value
- How many rows for a formula with 2, 3, 4, or n propositional variables?
- How many different truth tables are possible for 2 propositional variables?
- How many different formulae are possible for 2 propositional variables?

Logical Equivalence

- Check the contrapositive law ...

X	Y	$X \rightarrow Y$	$\neg X$	$\neg Y$	$\neg Y \rightarrow \neg X$
F	F	T	T	T	T
F	T	T	T	F	T
T	F	F	F	T	F
T	T	T	F	F	T

Logical Equivalence

- Check a De Morgan equivalence ….

X	Y		$X \wedge Y$	$\neg(X \wedge Y)$	$\neg X$	$\neg Y$	$\neg X \vee \neg Y$
F	F	|	F	T	T	T	T
F	T	|	F	T	T	F	T
T	F	|	F	T	F	T	T
T	T	|	T	F	F	F	F

More equivalences ...

- Distribution law

X	Y	Z	$X \wedge Y$	$X \wedge Z$	$(X \wedge Y) \vee (X \wedge Z)$	$Y \vee Z$	$X \wedge (Y \vee Z)$
F	F	F	F	F	F	F	F
F	F	T	F	F	F	T	F
F	T	F	F	F	F	T	F
F	T	T	F	F	F	T	F
T	F	F	F	F	F	F	F
T	F	T	F	T	T	T	T
T	T	F	T	F	T	T	T
T	T	T	T	T	T	T	T

Equivalence Example

- Are columns 5 and 7 equivalent?

X	Y	Z	$X \wedge Y$	$(X \wedge Y) \rightarrow Z$	$(Y \rightarrow Z)$	$X \rightarrow (Y \rightarrow Z)$
F	F	F	F	T	T	T
F	F	T	F	T	T	T
F	T	F	F	T	F	T
F	T	T	F			
T	F	F	F			
T	F	T	F			
T	T	F	T			
T	T	T	T			

Using an equivalence

(example from Discrete Maths and Combinatorics, Grimaldi)

```
z:=4;
For  i := 1 to 10 do
 Begin
 x := z - i;
 y := z + (3*i);
 If (x>0) and (y > 0) then
   Writeln('sum is', x+y)
 End;
```

```
z:=4;
For  i := 1 to 10 do
 Begin
 x := z - i;
 y := z + (3*i);
 If x>0 then
  If y > 0 then
    Writeln('sum is', x+y)
 End;
```

Laws of Logic

- Once it is shown by a truth table that the left-hand side is logically equivalent to the right-hand side … then whenever one meets the left-hand side in a formula, one can replace it by the right-hand side …
- Thus, a syntactic transformation Law of Logic has been produced

The following laws of logic can all be shown using truth tables to establish logical equivalence

Laws of Logic

- Double negation $\neg\neg p \Leftrightarrow p$
- DeMorgan's Laws $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$

  $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$

- Commutative Laws $p \vee q \Leftrightarrow q \vee p$

  $p \wedge q \Leftrightarrow q \wedge p$

- Associative Laws $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$

  $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$

Laws of Logic

- Distributive Laws $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

  $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

- Idempotent Laws $p \vee p \Leftrightarrow p$

  $p \wedge p \Leftrightarrow p$

- Identity Laws $p \vee F \Leftrightarrow p$

  $p \wedge T \Leftrightarrow p$

Laws of Logic

- Inverse Laws

$$p \vee \neg p \Leftrightarrow T$$

$$p \wedge \neg p \Leftrightarrow F$$

- Domination Laws

$$p \vee T \Leftrightarrow T$$

$$p \wedge F \Leftrightarrow F$$

- Absorption Laws

$$p \vee (p \wedge q) \Leftrightarrow p$$

$$p \wedge (p \vee q) \Leftrightarrow p$$

Additional Laws

- Implication Defn. $p \to q \Leftrightarrow \neg p \vee q$
- If-and-only-if Defn, $p \leftrightarrow q \Leftrightarrow (p \to q) \wedge (q \to p)$

- Truth tables provide definitions for ¬,Λ,V etc
- ...

Using Laws of Logic

Simplification:

- Algebraic laws allow us to transform formulae, for example, to simplify them
- if a formula simplifies to T
  - then it's a tautology
- if a formula simplifies to F
  - then it's a contradiction

Simplification

- If a process is not suspended and the process is either active or suspended then it is active.

$$\neg suspended \wedge (active \vee suspended) \rightarrow active$$

[fm] ¬suspended ⋀ (suspended ⋁ active)--> active

[1] ¬(¬suspended ⋀ (suspended ⋁ active)) ⋁ active [fm, Implication Defn]

[2] ¬¬suspended ⋁ ¬(suspended ⋁ active) ⋁ active [1,DeMorgan Law]

[3] suspended ⋁ ¬(suspended ⋁ active) ⋁ active [2,Double Negation]

[4] suspended ⋁ (¬suspended ⋀ ¬active) ⋁ active [3, DeMorgan Law]

[5] (suspended ⋁ ¬suspended) ⋀ (suspended ⋁ ¬active) ⋁ active [4, Distrib Law]

[6] (T ⋀ (suspended ⋁ ¬active)) ⋁ active [5, Inverse Law]

[7] suspended ⋁ ¬active ⋁ active [6, Domination, Associativity Law]

[7] suspended ⋁ T [7, Inverse Law]

[8] T [7, Domination Law]

Simplification

[fm] (idle --> dataready) $\wedge$ (idle $\vee$ working) $\wedge$ ¬dataready --> working **?**

[1]

[2]

[3]

[4]

[5]

[6]

[7]

[7]

[8]

Normal Form

- Rather than transforming formulae in an ad-hoc fashion using the rules, can put formulae into a standardized or "normalized" form.

Examples: conjunctive normal form(CNF), and disjunctive normal form(DNF)

- A statement is in conjunctive normal form(CNF) if it is a conjunction (AND) of the disjunction (OR) of one or more literals.
- A literal is an atomic formula (e.g. statement letter A) or the negation of an atomic formula.

Normal Forms

Example conjunctive normal form (CNF):

$$(A \lor \neg B) \land (C \lor \neg A) \land (\neg B \lor C \lor D)$$

Example disjunctive normal form(DNF):

$$(X \land Y) \lor (A \land \neg X \land Y) \lor (\neg A \land X)$$

- Equivalence rules can be used to translate any formula into a regular normal form, CNF or DNF

Conjunctive Normal Form

- Which are in CNF?

  $(A \lor \neg B) \land (C \lor \neg A) \land (\neg B \lor C)$

  $A \lor B$

  $A \land C$

  A

  $(D \lor \neg B) \lor (C \lor \neg A)$

  $(A \lor \neg B) \land \neg(C \lor \neg A)$

Conversion to CNF

- Replace A $\leftarrow\rightarrow$ B by (A $\rightarrow$ B) $\wedge$ (B $\rightarrow$ A)
- Eliminate $\rightarrow$ by replacing A $\rightarrow$ B with $\neg$A $\vee$ B
- Puch $\neg$ in using double negative and De Morgan's rules:

  $\neg$ ($\neg$ A) $\Leftrightarrow$ A

  $\neg$ (A $\wedge$ B) $\Leftrightarrow$ ($\neg$ A $\vee$ $\neg$B)

  $\neg$ (A $\vee$ B) $\Leftrightarrow$ ($\neg$ A $\wedge$ $\neg$B)
- Use distributive laws:

  A $\vee$ (B $\wedge$ C) $\Leftrightarrow$ (A $\vee$ B) $\wedge$ (A $\vee$ C)

  (A $\vee$ B) $\wedge$ C) $\Leftrightarrow$ (A $\wedge$ C) $\vee$ (B $\wedge$ C)

Conversion to CNF

Exercise, convert the following to CNF:

$(A \rightarrow B) \rightarrow (\neg C \rightarrow (B \wedge C))$

…

…

Simplify the CNF

- A disjunct with A and ¬A reduces to T
- Delete any disjunct that includes another disjunct e.g. $A \vee B \vee C$ includes $A \vee B$
- Replace disjuncts $A \vee B$ and $\neg A \vee B$, by B

… this also means disjuncts A and ¬ A reduce to F (putting B = F in the rule)