# Overview of Graph Theory

## for networking applications

Gregory Provan

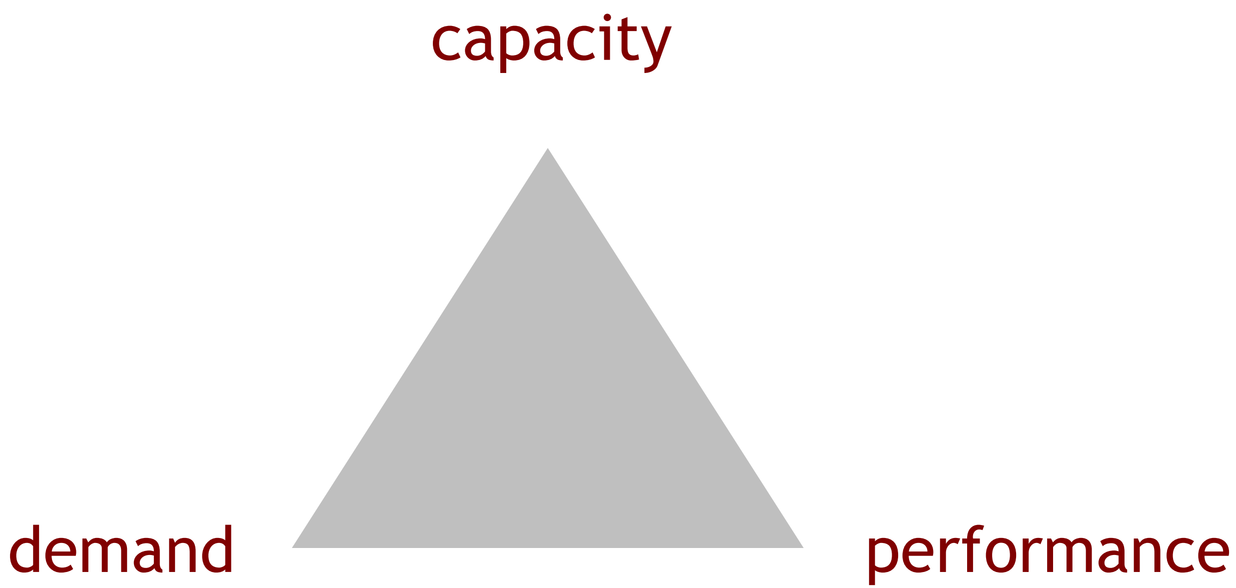
# Introduction to Probability Theory

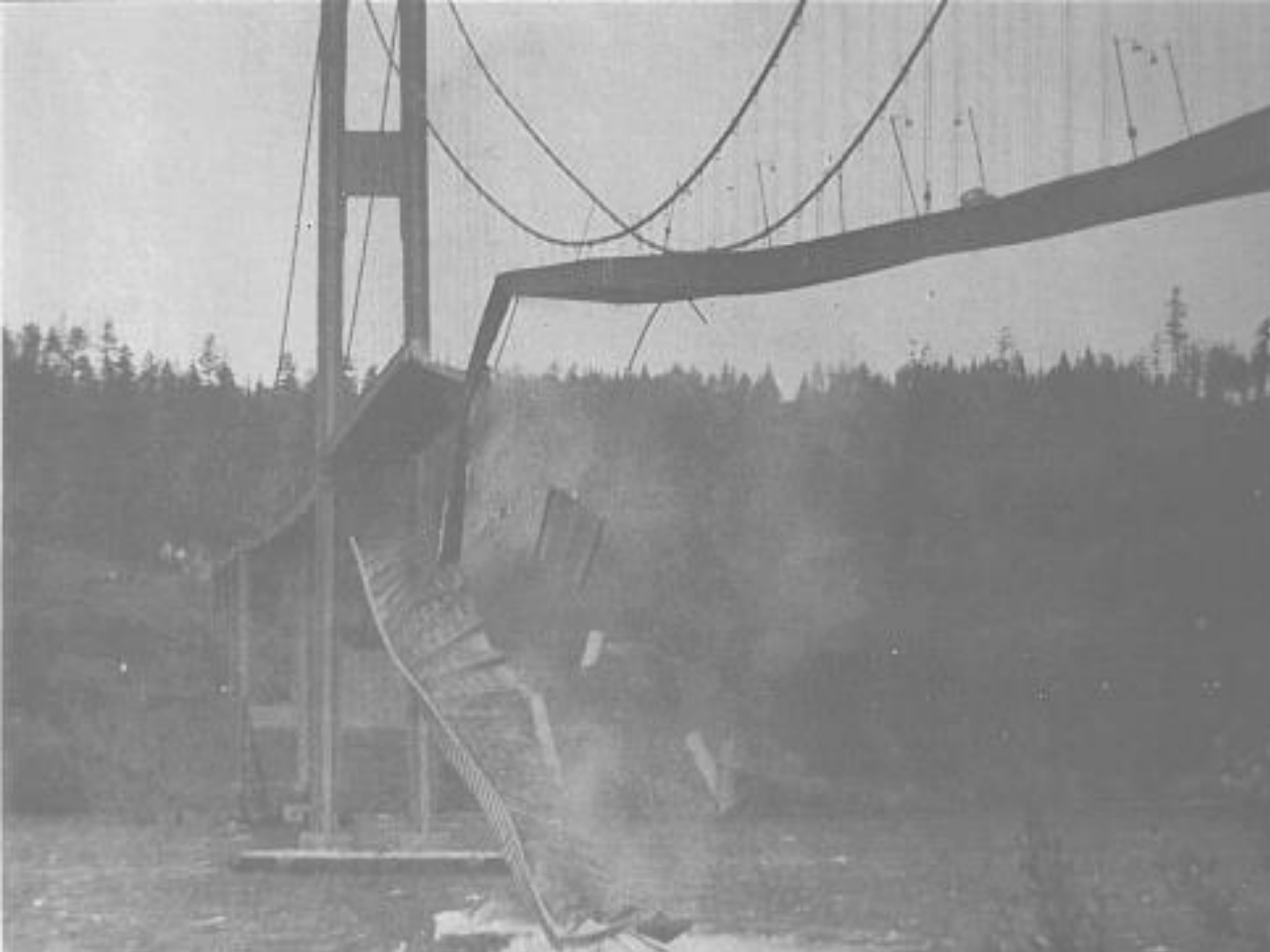
# Probability Theory: Outline

- Motivation
- Basic concepts in probability theory
- Bayes' rule
- Random variables and distributions

# The role of the engineer

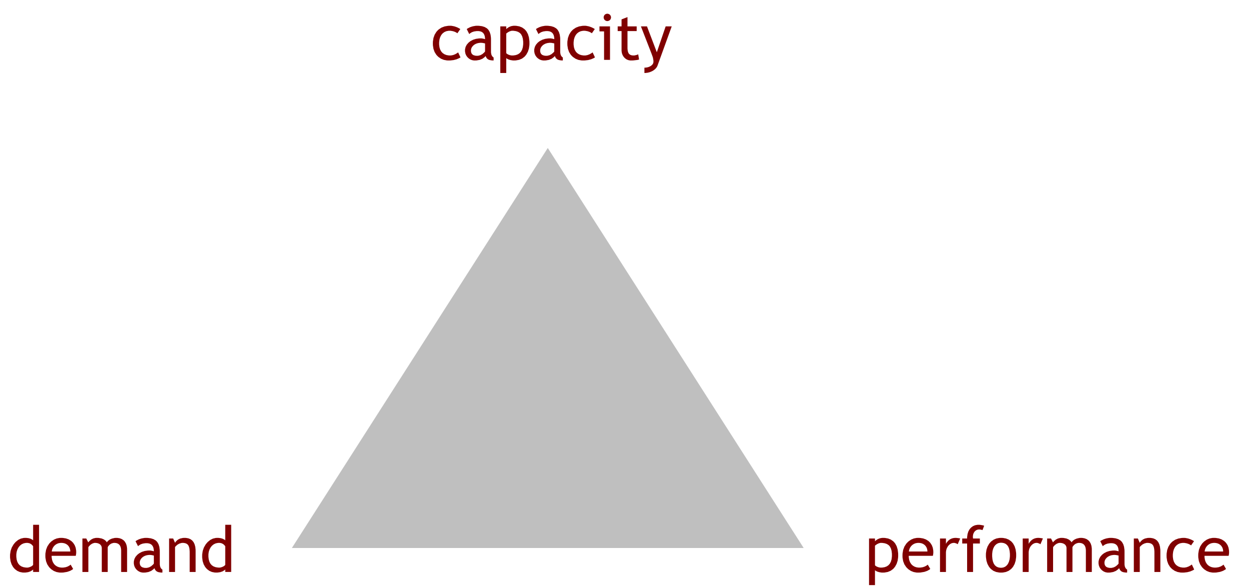
- to quantify the three-way relationship between demand, capacity and performance





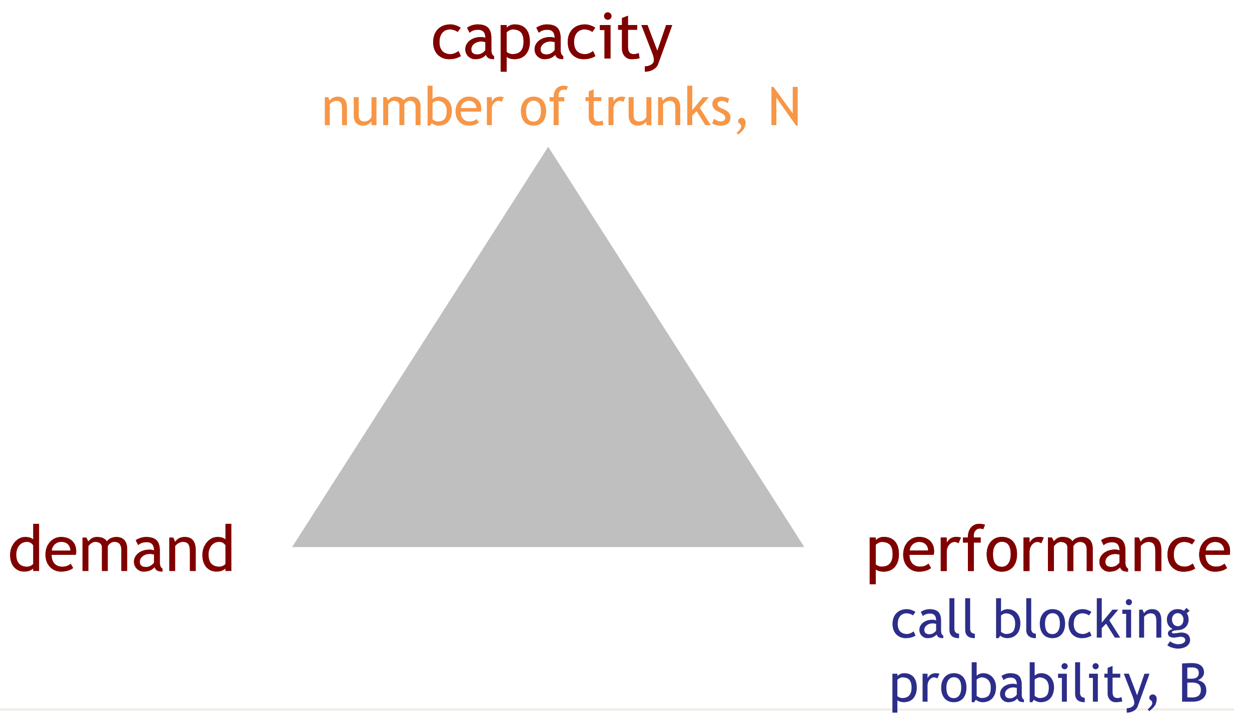
# The role of the network engineer

- to quantify the three-way relationship between demand, capacity and performance

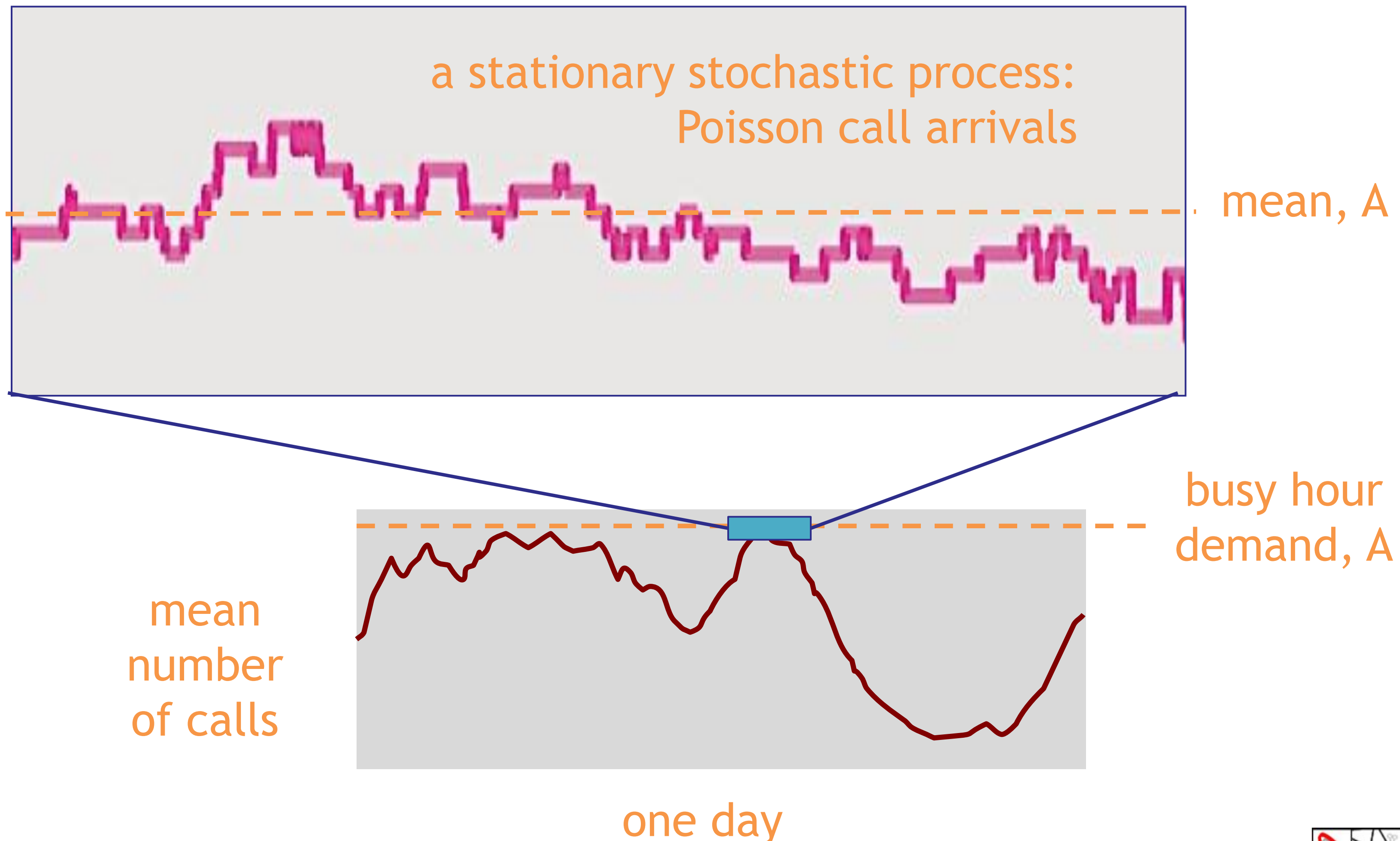


# The role of the network engineer

- an example from the telephone network: the Erlang formula



# Traffic variations and stationarity



# The role of the network engineer

- an example from the telephone network: the Erlang formula
- insensitivity (of performance to detailed traffic characteristics) facilitates engineering

**capacity**

number of trunks, N

$$B = \frac{A^N / N!}{\sum_{i=0}^{N} A^i / i!}$$

**demand**

Poisson call process of intensity A

**performance**

call blocking probability, B

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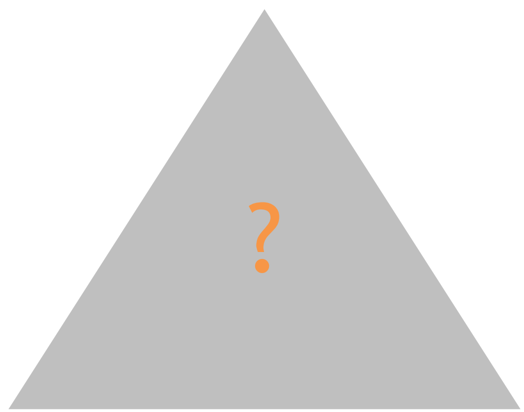
# The role of the network engineer

- what about the Internet? what about the Cloud?

**capacity**
bandwidth and how it is shared




**demand**
a stationary
arrival process...

**performance**
loss rates,
response times,...

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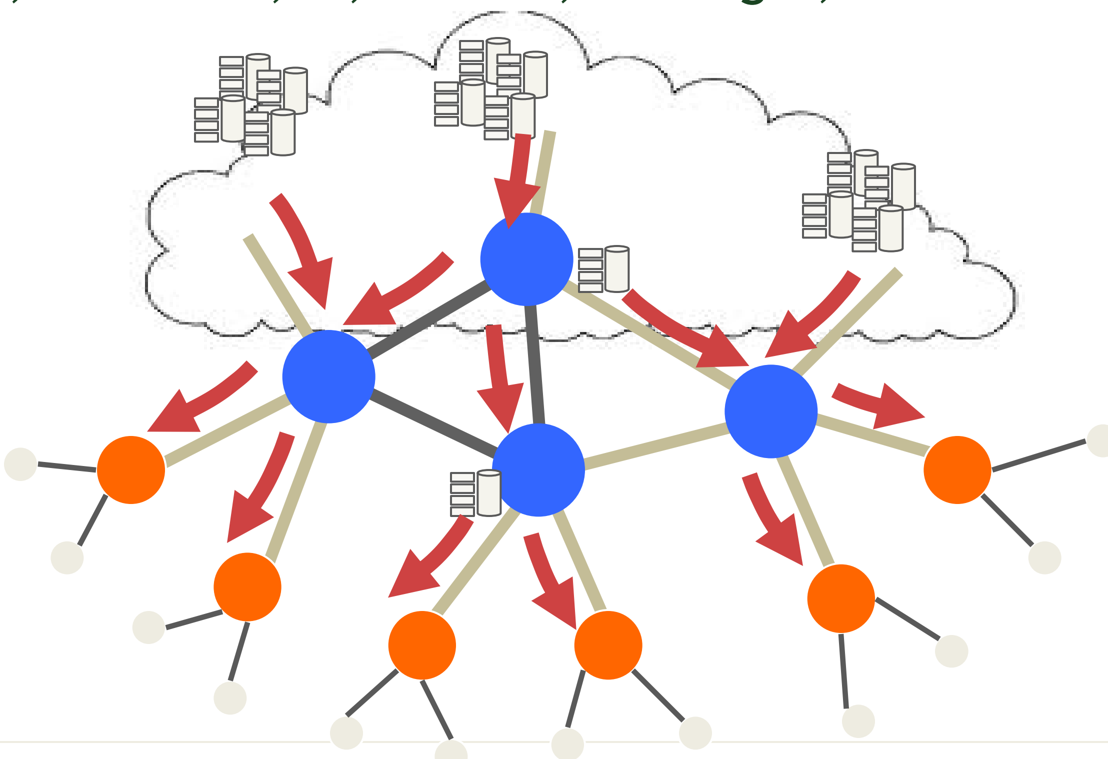
# A network of data centers

- most traffic in an ISP network originates in a data center
  - Google, Facebook,..., Akamai, Limelight,...

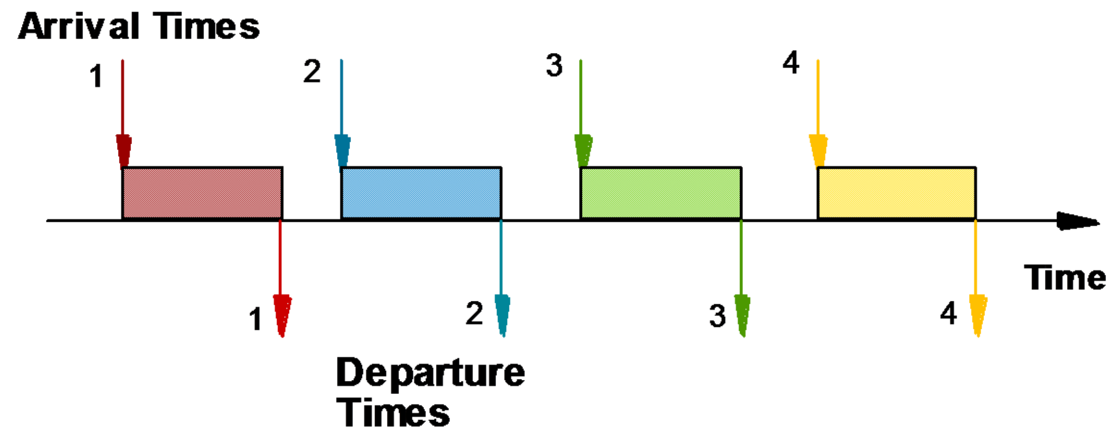


# Probability Theory: Outline

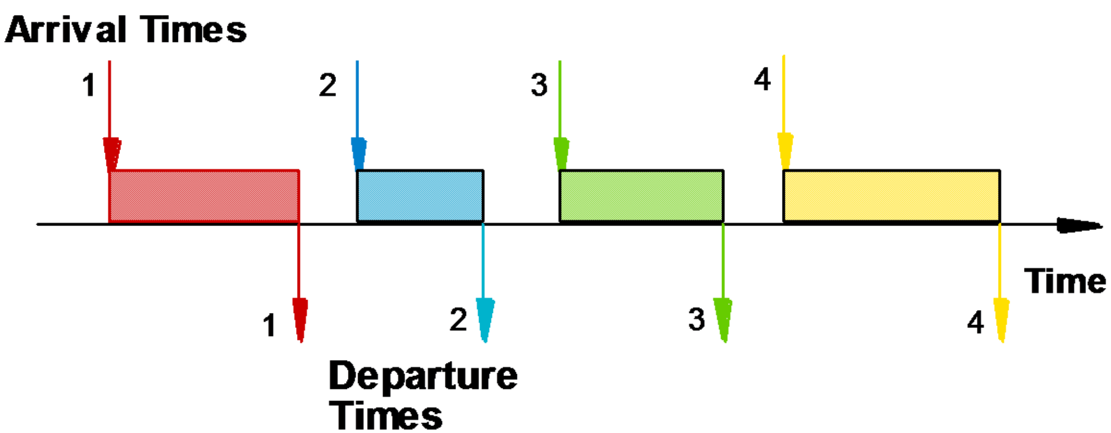
- Motivation
- Basic concepts in probability theory
- Bayes' rule
- Random variables and distributions

# Stochastic Packet Distribution

- Packets may arrive randomly, and have random length



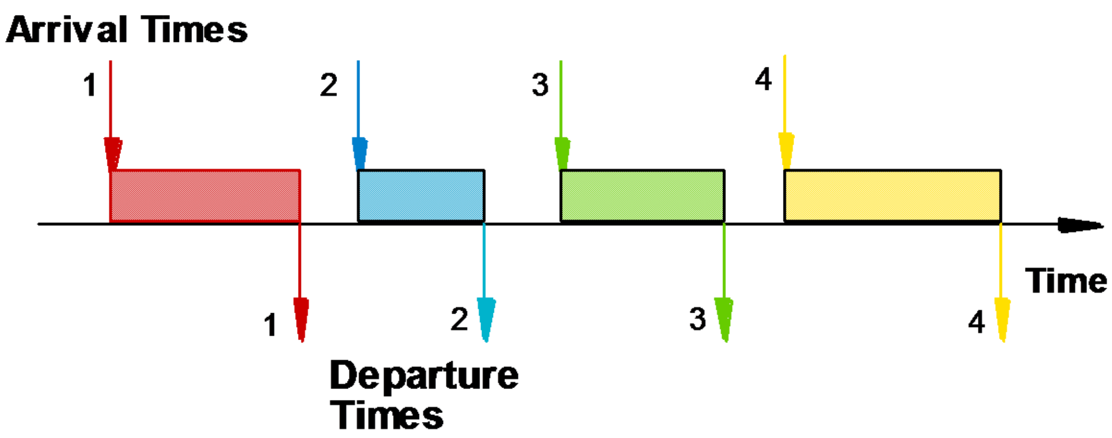

Regular Traffic




Irregular but Spaced Apart Traffic

# Probabilistic Models

- P(arrival=t)
- P(packet length = $\tau$)
- P(departure = t+$\tau$)



# Definition of Probability

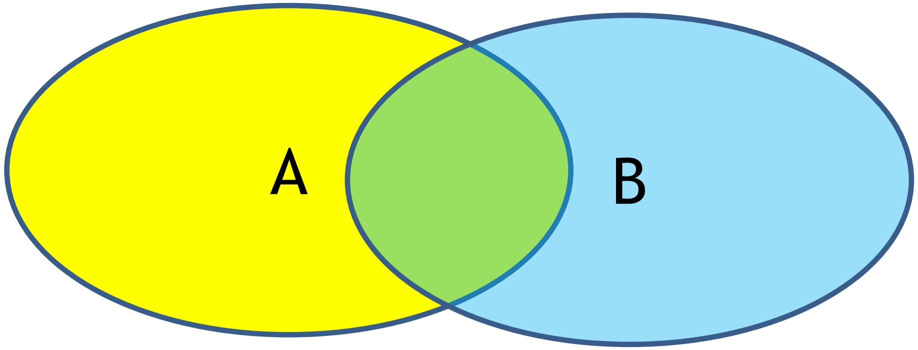
- ***Experiment***: toss a coin twice
- ***Sample space***: possible outcomes of an experiment
  - S = {HH, HT, TH, TT}
- ***Event***: a subset of possible outcomes
  - A={HH}, B={HT, TH}
- ***Probability of an event*** : a measure assigned to an event A, Pr(A)
  - Axiom 1: Pr(A) $\geq$ 0
  - Axiom 2: Pr(S) = 1
  - Axiom 3: For every sequence of disjoint events

$$\Pr(\bigcup_i A_i) = \sum_i \Pr(A_i)$$

- Example: Pr(A) = n(A)/N

# Joint Probability

- For events A and B, the **joint probability** Pr(AB) stands for the probability that both events happen.
- Example: A={HH}, B={HT, TH}, what is the joint probability Pr(AB)?



# Independence

- Two events ***A and B are independent*** in case

$$\Pr(AB) = \Pr(A)\Pr(B)$$

- A set of events $\{A_i\}$ is independent iff

$$\Pr\left(\bigcap_i A_i\right) = \prod_i \Pr(A_i)$$

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- Example: Drug test

A = {A patient is a Women}

B = {Drug fails}

Will event A be independent from event B ?

| | Women | Men |
|---|---|---|
| Success | 200 | 1800 |
| Failure | 1800 | 200 |

# Independence

- Consider the experiment of tossing a coin twice
- Example I:
  - A = {HT, HH}, B = {HT}
  - Is event A independent from event B?
- Example II:
  - A = {HT}, B = {TH}
  - Is event A independent from event B?
- Disjoint $\neq$ Independence

- If A is independent from B, B is independent from C, will A be independent from C?

# Conditioning

- If A and B are events with Pr(A) > 0, the ***conditional probability of B given A*** is

$$\Pr(B \mid A) = \frac{\Pr(AB)}{\Pr(A)}$$

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A = {Patient is a Woman}

B = {Drug fails}

Pr(B|A) = ?

Pr(A|B) = ?

# Conditioning

- If A and B are events with Pr(A) > 0, the ***conditional probability of B given A*** is

$$\Pr(B \mid A) = \frac{\Pr(AB)}{\Pr(A)}$$

- Example: Drug test

| Pr(outcome \| gender) | Women | Men |
|---|---|---|
| Success | 200 | 1800 |
| Failure | 1800 | 200 |

A = {Patient is a Women}

B = {Drug fails}

Pr(B|A) = ?

Pr(A|B) = ?

- Given A is independent from B, what is the relationship between Pr(A|B) and Pr(A)?

# Conditional Independence

- Event A and B are ***conditionally independent given C*** in case

  $$\Pr(AB|C)=\Pr(A|C)\Pr(B|C)$$

- A set of events $\{A_i\}$ is conditionally independent given C in case

  $$\Pr(\bigcup_i A_i \mid C) = \prod_i \Pr(A_i \mid C)$$

# Conditional Independence (cont'd)

- Example: There are three events: A, B, C
  - $Pr(A) = Pr(B) = Pr(C) = 1/5$
  - $Pr(A,C) = Pr(B,C) = 1/25$, $Pr(A,B) = 1/10$
  - $Pr(A,B,C) = 1/125$
  - Are A, B are independent?
  - Are A, B are conditionally independent given C?
- A and B are independent $\neq$ A and B are conditionally independent

# Outline

- Important concepts in probability theory
- Bayes' rule
- Random variables and distributions

# Bayes' Rule

- Given two events A and B and suppose that Pr(A) > 0. Then

$$\Pr(B \mid A) = \frac{\Pr(AB)}{\Pr(A)} = \frac{\Pr(A \mid B)\Pr(B)}{\Pr(A)}$$

- Example:

$\Pr(R) = 0.8$

| $\Pr(W \mid R)$ | $R$ | $\neg R$ |
|---|---|---|
| $W$ | 0.7 | 0.4 |
| $\neg W$ | 0.3 | 0.6 |

R: It is a rainy day

W: The grass is wet

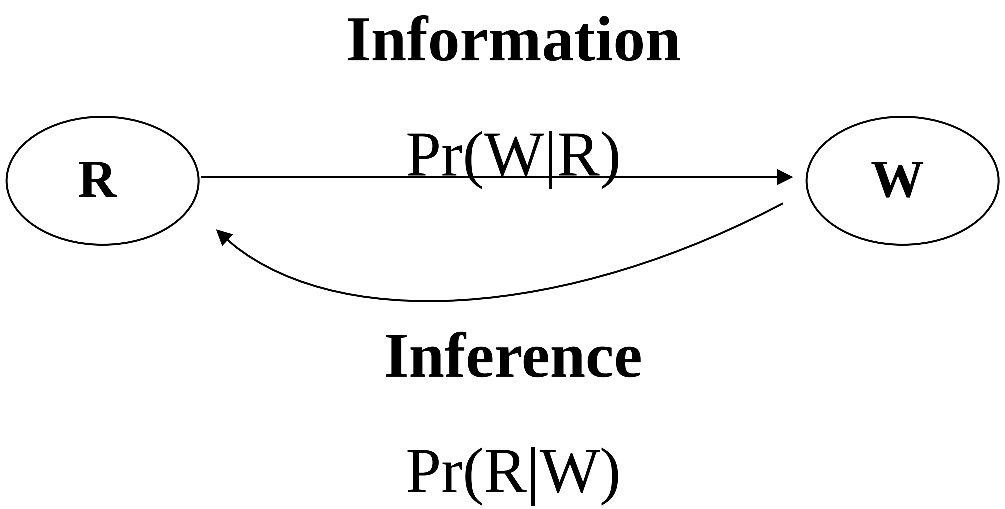
$\Pr(R \mid W) = ?$

# Bayes' Rule

| | R | ¬R |
|---|---|---|
| W | 0.7 | 0.4 |
| ¬W | 0.3 | 0.6 |

R: It rains

W: The grass is wet



# Bayes' Rule

| | R | ¬R |
|---|---|---|
| W | 0.7 | 0.4 |
| ¬W | 0.3 | 0.6 |

R: It rains

W: The grass is wet

**Information:**

**Posterior** **Likelihood** **Prior**

$$\boxed{\Pr(H \mid E)} = \frac{\boxed{\Pr(E \mid H)}\boxed{\Pr(H)}}{\Pr(E)}$$

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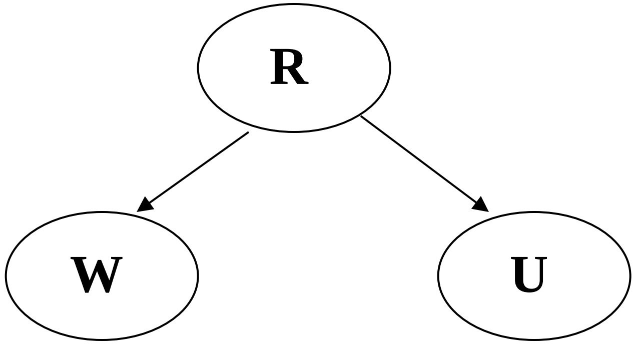
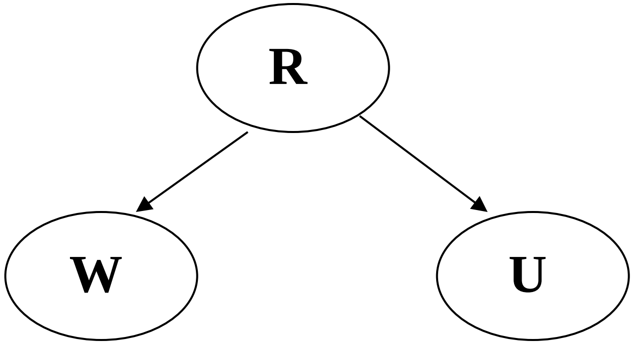
# A More Complicated Example




| | |
|---|---|
| **R** | It rains |
| **W** | The grass is wet |
| **U** | People bring umbrella |

# A More Complicated Example




**R** It rains

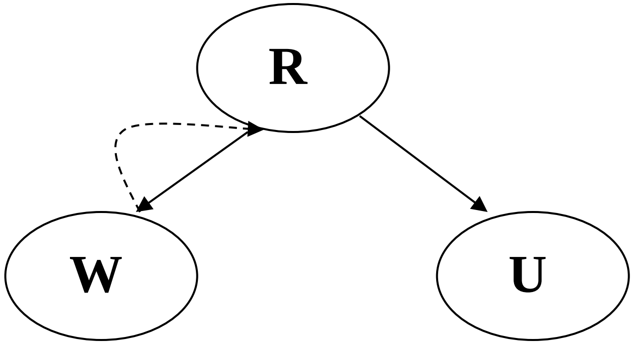
**W** The grass is wet

**U** People bring umbrella

Pr(UW|R)=Pr(U|R)Pr(W|R)

Pr(UW| ¬R)=Pr(U| ¬R)Pr(W| ¬R)

# A More Complicated Example




$\Pr(R) = 0.8$

**R** It rains

**W** The grass is wet

**U** People bring umbrella

$\Pr(UW|R)=\Pr(U|R)\Pr(W|R)$

$\Pr(UW|\neg R)=\Pr(U|\neg R)\Pr(W|\neg R)$

| Pr(W\|R) | R | ¬R |
|---|---|---|
| W | 0.7 | 0.4 |
| ¬W | 0.3 | 0.6 |

| Pr(U\|R) | R | ¬R |
|---|---|---|
| U | 0.9 | 0.2 |
| ¬U | 0.1 | 0.8 |

$\Pr(U|W) = ?$

# Outline

- Important concepts in probability theory
- Bayes' rule
- Random variable and probability distribution

# Random Variable and Distribution

- A ***random variable X*** is a numerical outcome of a random experiment
- The ***distribution*** of a random variable is the collection of possible outcomes along with their probabilities:
  - Discrete case: $\Pr(X = x) = p_\theta(x)$
  - Continuous case: $\Pr(a \le X \le b) = \int_a^b p_\theta(x)dx$

# Random Variable: Example

- Let S be the set of all sequences of three rolls of a die. Let X be the sum of the number of dots on the three rolls.
- What are the possible values for X?
- Pr(X = 5) = ?, Pr(X = 10) = ?

# Expectation

- A random variable X~Pr(X=x). Then, its expectation is

$$E[X] = \sum_x x \Pr(X = x)$$

  - In an empirical sample, x1, x2,…, xN,

$$E[X] = \frac{1}{N} \sum_{i=1}^{N} x_i$$

- Continuous case: $E[X] = \int_{-\infty}^{\infty} x p_\theta(x) dx$

- Expectation of sum of random variables

$$E[X_1 + X_2] = E[X_1] + E[X_2]$$

# Expectation: Example

- Let S be the set of all sequence of three rolls of a die. Let X be the sum of the number of dots on the three rolls.
- What is E(X)?

- Let S be the set of all sequence of three rolls of a die. Let X be the product of the number of dots on the three rolls.
- What is E(X)?

# Variance

- The variance of a random variable X is the expectation of $(X\text{-}E[x])^2$ :

$$
\begin{aligned}
Var(X) &= E((X - E[X])^2) \\
&= E(X^2 + E[X]^2 - 2XE[X]) \\
&= E(X^2 - E[X]^2) \\
&= E[X^2] - E[X]^2
\end{aligned}
$$

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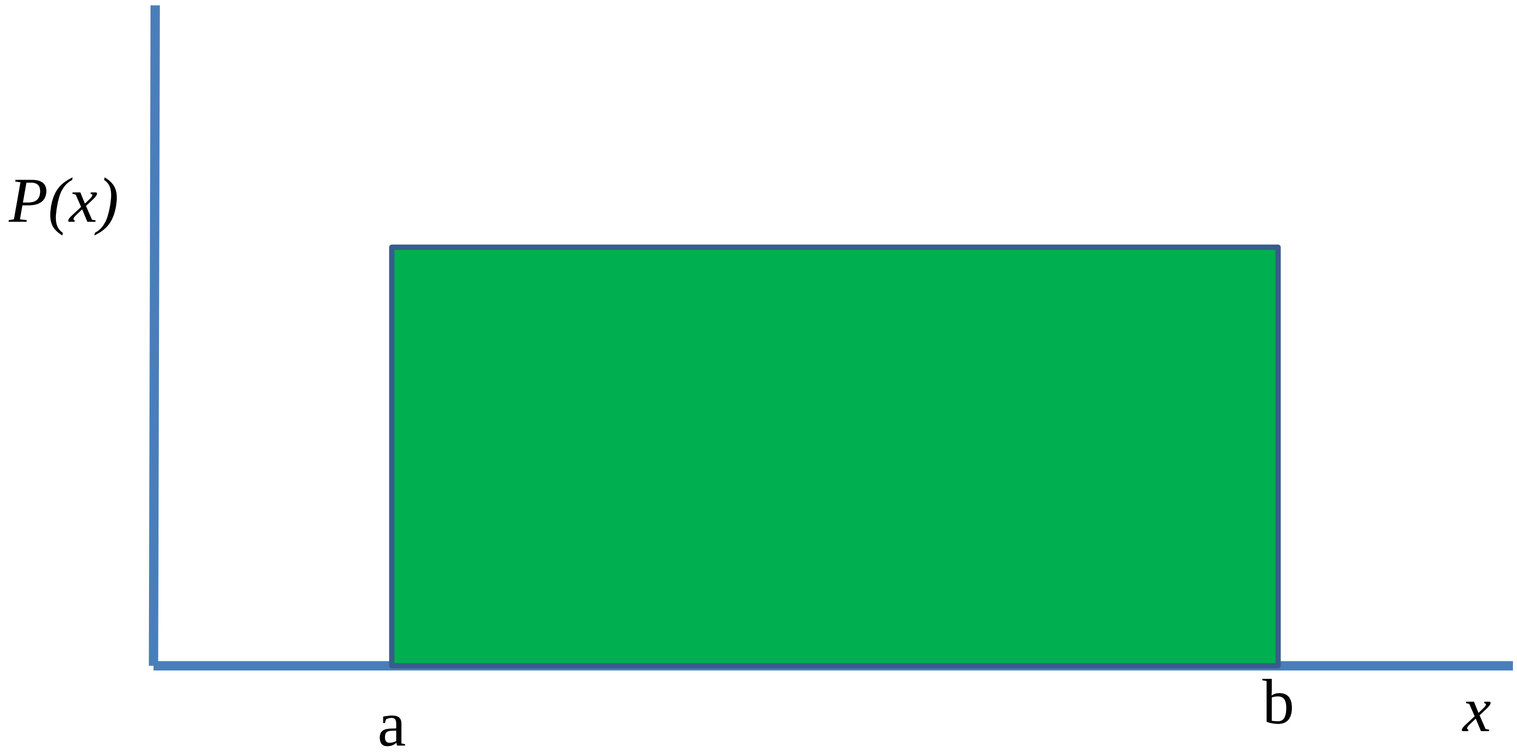
# Important Distributions

- Uniform distribution
- Bernoulli distribution
  - Binary outcomes
- Poisson distribution
- Normal (Gaussian) distribution

# Uniform Distribution

- Probability is constant over a range (a,b)



# Bernoulli Distribution

- The outcome of an experiment can either be *success* (i.e., 1) or *failure* (i.e., 0).
- Pr(X=1) = p, Pr(X=0) = 1-p, or

$$p_\theta(x) = p^x (1-p)^{1-x}$$

- E[X] = p, Var(X) = p(1-p)

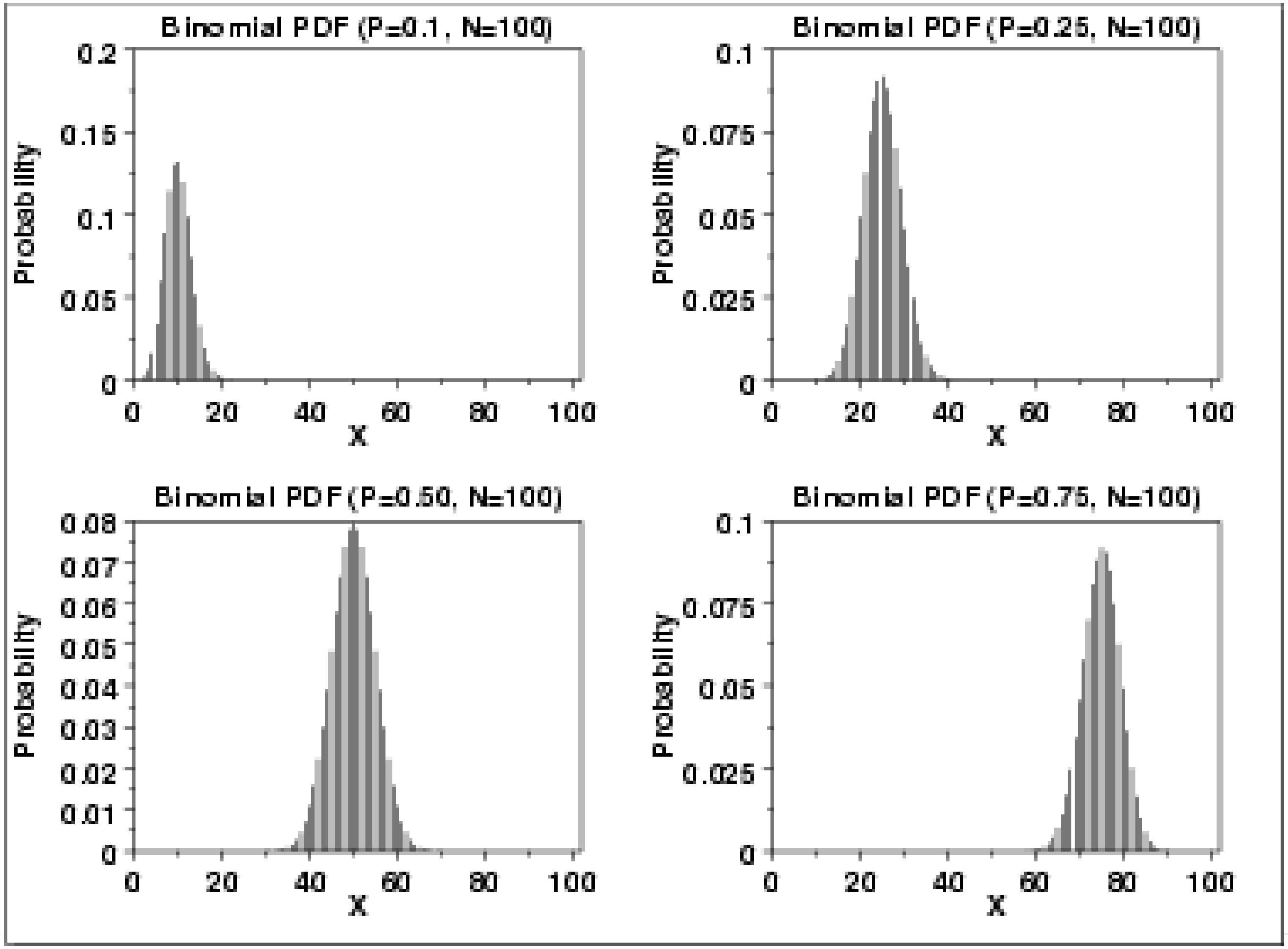
# Binomial Distribution

- n draws of a Bernoulli distribution
  - $X_i \sim \text{Bernoulli}(p)$, $X=\sum_{i=1}^{n} X_i$, $X \sim \text{Bin}(p, n)$
- Random variable X stands for the number of times that experiments are successful.

$$\Pr(X = x) = p_\theta(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x = 1, 2, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

- E[X] = np, Var(X) = np(1-p)

# Plots of Binomial Distribution



# Poisson Distribution

- Coming from Binomial distribution
  - Fix the expectation $\lambda$=np
  - Let the number of trials n$\rightarrow\infty$

  A Binomial distribution will become a Poisson distribution

$$\Pr(X = x) = p_\theta(x) = \begin{cases} \dfrac{\lambda^x}{x!} e^{-\lambda} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

- E[X] = $\lambda$, Var(X) = $\lambda$

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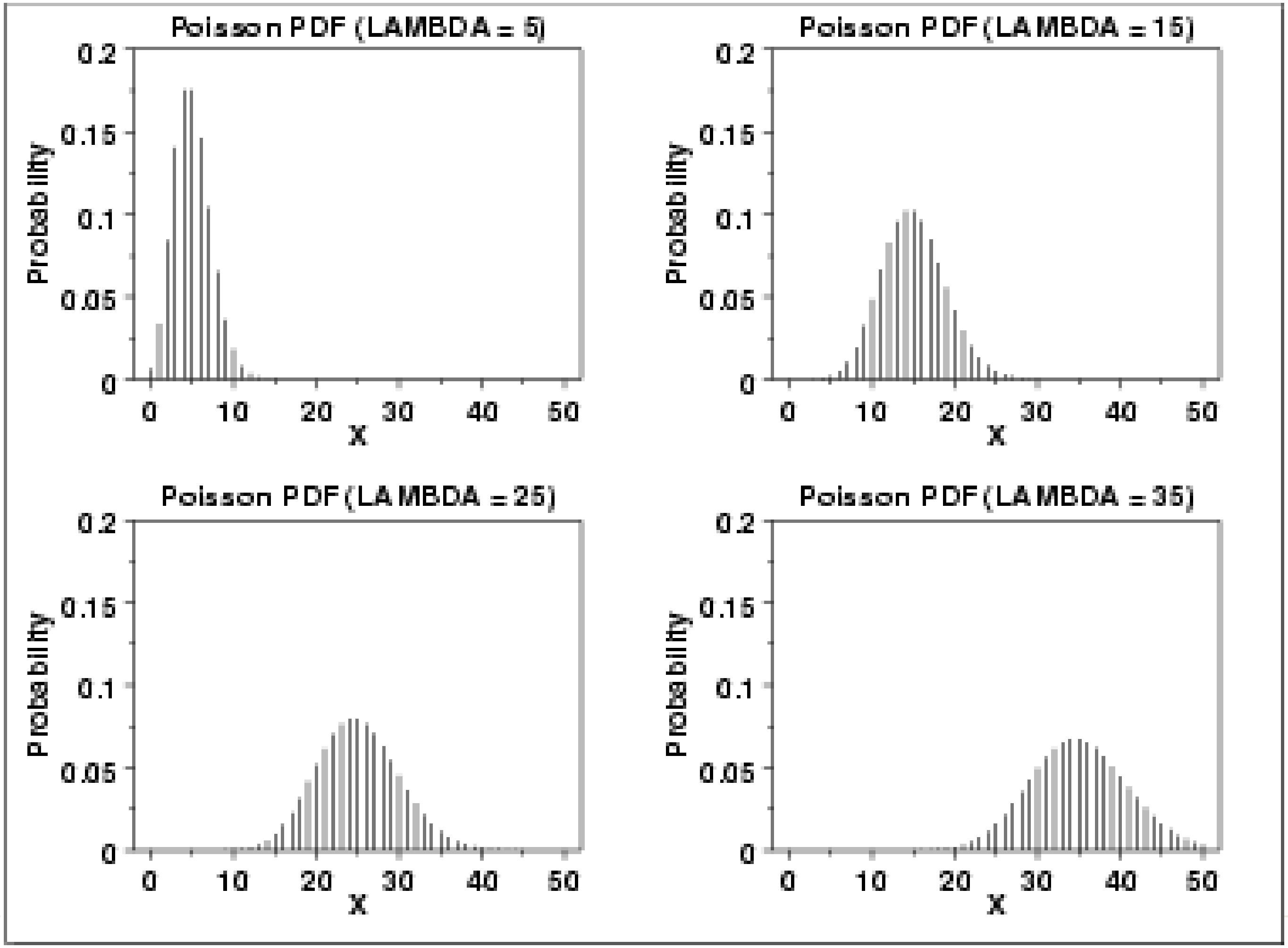
# Plots of Poisson Distribution



# Normal (Gaussian) Distribution

- $X \sim N(\mu, \sigma)$

$$p_\theta(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$$

$$\Pr(a \le X \le b) = \int_a^b p_\theta(x)dx = \int_a^b \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx$$

- $E[X] = \mu$, $Var(X) = \sigma^2$
- If $X_1 \sim N(\mu_1, \sigma_1)$ and $X_2 \sim N(\mu_2, \sigma_2)$, $X = X_1 + X_2$ ?

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