CS2205: Theory of Computation
Problem Class I (Set Theory)

A **set** is a collection of objects. The objects in a set are called its **members** or **elements**.

There are two ways to define a set:

Extensional definition: *enumerate* the members E.g. $\{5, 7, 10, 103\}$

Intensional definition: specify a property, called the **defining property**, possessed by all the members and no other objects. E.g.

$$\{x \mid x \text{ is an integer and } x \geq 0 \text{ and } x < 4\}$$

$$\{l \mid l \text{ is a lecturer in the Comp. Sci. Dept. at UCC}\}$$

Two fundamental properties of sets:

1. Each element occurs only once.
2. The elements have no order.

For example, the following different expressions all denote the same set:

$$\begin{array}{ccc} \{2,3,4\} & \{3,2,4\} & \{2,2,3,4,4,4\} \\ \{n \mid 2 \leq n \leq 4\} & \{m \mid 2 \leq m \leq 4\} & \{n \mid 1 < n < 5\} \end{array}$$

(assuming n and m only range over integers in the last three).

The number of members of a set A is called the **cardinality** of A. The cardinality of set A will be denoted $|A|$. For example, $|\{0,1,2\}| = 3$ and $|\{3\}| = 1$. (Sets whose cardinality is 1 are often called **singleton sets**.)

Exercise 1.

1. Give *extensional* definitions for each of the following:
   (a) The set of all positive even integers that are less than 3.
   (b) $\{x \mid x/3 \text{ is a positive integer and } x < 15\}$

2. Give *intensional* definitions for each of the following. (There are many possible answers.)
   (a) The set of all positive even integers.
   (b) $\{14, 12, 10, 8, 6, 4, 2\}$
   (c) $\{3, 3, 4, 4, 5, 5, 6, 6\}$
3. Which of the following statements are true and which are false?
   (a) $\{5, 10, 15\} = \{x \mid x/5 \text{ is a positive integer and } 5 < x < 15\}$
   (b) $\{1, 2, 2, 3, 3, 3\} = \{1, 1, 2, 2, 3, 3\}$
   (c) $\{25, 12, 2004\} \neq \{12, 25, 2004\}$
4. Evaluate the following:
   (a) $|\{31, 28, 31, 30, 31, 30, 31, 31, 30, 31, 31, 31\}|$
   (b) $|\{5\}| = |\{x \mid x \text{ is an integer and } 1 \leq x \leq 5\}|$

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Names: We can also give names to sets using $=_{\text{def}}$, and then use the names instead of the extensional or intensional definitions. E.g. *DGB_Modules* $=_{\text{def}}$ {*CS2205, CS4401, CS4409*}

Some names for sets are already in common use:

$\mathbb{Z}$	the **integers** (i.e. all 'whole' numbers)
$\mathbb{N}$	the **natural numbers** (i.e. all integers ≥ 0)
$\mathbb{P}$	the **positive integers** (i.e. all integers > 0)
$\mathbb{Q}$	the **rationals**
$\mathbb{R}$	the **reals**

The **universal set**, usually written U, is the set of all elements of interest to your present purpose.

The **empty set** has no elements. It is usually written $\{\}$ or $\emptyset$. Its has zero cardinality: $|\emptyset| = 0$.

Set membership: For any set A and any object a, either a is a member of A or it is not.

- To indicate that a is a member of A, we write $a \in A$
- To indicate that a is not a member of A, we write $a \notin A$

E.g.

- $1 \in \{0, 1, 2\}$
- *Cork* $\notin \{c \mid c$ is a capital city$\}$

Intensional set definitions revisited: It is very common for an intensional set definition to take the following form:

$$\{x \mid x \in A \text{ and } P(x)\}$$

(This set will be members of A that satisfy property P.) E.g.

$$\{n \mid n \in \mathbb{N} \text{ and } 1 < n < 10\}$$

For this common form of definition, a shorthand is popular:

$$\{x \in A \mid P(x)\}$$

E.g.

$$\{n \in \mathbb{N} \mid 1 < n < 10\}$$

Subset: For any two sets A_1 and A_2, A_1 is a **subset** of A_2 if and only if for all a, if $a \in A_1$ then $a \in A_2$.

- To indicate that A_1 is a subset of A_2, we write $A_1 \subseteq A_2$
- To indicate that A_1 is not a subset of A_2, we write $A_1 \not\subseteq A_2$

E.g.

- $\{0, 2\} \subseteq \{0, 1, 2\}$
- $\{0, 1, 3\} \not\subseteq \{0, 1, 2\}$
- $\{0, 1, 2\} \subseteq \{0, 1, 2\}$

Proper subset: For any two sets A_1 and A_2, A_1 is a **proper subset** of A_2 if and only if $A_1 \subseteq A_2$ and there exists a such that $a \in A_2$ but $a \notin A_1$. This is written $A_1 \subset A_2$. E.g.

- $\{0, 2\} \subset \{0, 1, 2\}$
- $\{0, 1, 3\} \not\subset \{0, 1, 2\}$
- $\{0, 1, 2\} \not\subset \{0, 1, 2\}$
- $\mathbb{P} \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$

Superset: $A_1 \supseteq A_2 =_{\text{def}} A_2 \subseteq A_1$

Proper superset: $A_1 \supset A_2 =_{\text{def}} A_2 \subset A_1$

E.g.

- $\{0, 1, 2\} \supseteq \{0, 1\}$
- $\{0, 1, 2\} \supset \{0, 1\}$

From the definitions, the following should be apparent:

- For any set A, $A \subseteq A$.
- For any set A, $\emptyset \subseteq A$.

Exercise 2. $U =_{\text{def}} \{x \in \mathbb{P} \mid x \leq 10\}$
$S1 =_{\text{def}} \{3, 6, 9\}$
$S2 =_{\text{def}} \{2, 4, 6, 8, 10\}$

1. Write down the cardinalities of each of the sets $U, S1$, and $S2$.
2. Rewrite each of the following expressions, replacing the dots $(\ldots)$ with $\in, \subseteq$ and/or $\subset$ to make true statements. (Sometimes there will be more than one answer, in which case give them all. Sometimes there will be no answer, in which case write 'None'.)

   Example: for $S1 \ldots U$ the answers are $S1 \subseteq U$ and $S1 \subset U$.

   (a) $3 \ldots S1$
   (b) $\{3\} \ldots S1$
   (c) $\{3, 9\} \ldots S1$
   (d) $S2 \ldots U$
   (e) $S1 \ldots S2$
   (f) $S2 \ldots S2$
   (g) $\emptyset \ldots U$
   (h) $\emptyset \ldots S1$

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The **union** of two sets A_1 and A_2 is the set denoted by $A_1 \cup A_2$ and defined as follows:

$$A_1 \cup A_2 =_{\text{def}} \{a \mid a \in A_1 \text{ or } a \in A_2\}$$

E.g.

- $\{0, 1, 2\} \cup \{3, 4\} = \{0, 1, 2, 3, 4\}$
- $\{0, 1, 2\} \cup \{2, 3, 4\} = \{0, 1, 2, 3, 4\}$

The **intersection** of two sets A_1 and A_2 is the set denoted by $A_1 \cap A_2$ and defined as follows:

$$A_1 \cap A_2 =_{\text{def}} \{a \mid a \in A_1 \text{ and } a \in A_2\}$$

E.g.

- $\{0,1,2\} \cap \{1,2,3\} = \{1,2\}$
- $\{0,1\} \cap \{2,3,4\} = \emptyset$

Sets A_1 and A_2 are said to be **disjoint** if they have no element in common, i.e. $A_1 \cap A_2 = \emptyset$.

The **difference** of two sets A_1 and A_2 is the set denoted by $A_1 \setminus A_2$ and defined as follows:

$$A_1 \setminus A_2 =_{\text{def}} \{a \mid a \in A_1 \text{ and } a \notin A_2\}$$

E.g.

- $\{0,1,2,3\} \setminus \{1,2\} = \{0,3\}$
- $\{0,1,2,3\} \setminus \{2,3,4\} = \{0,1\}$
- $\{0,1,2\} \setminus \{3,4\} = \{0,1,2\}$

The **complement** of a set A is the set denoted by A' and defined as follows:

$$A' =_{\text{def}} \{a \mid a \in U \text{ and } a \notin A\}$$

E.g., if $U =_{\text{def}} \{0,1,2,3,4\}$, then $\{0,1,3\}' = \{2,4\}$.

Exercise 3.

1. If $U =_{\text{def}} \{n \in \mathbb{N} \mid n > 4\}$, evaluate $(\{5,6,7\} \setminus \{7,8\}) \subseteq \{x \in \mathbb{N} \mid x > 10\}'$
2. $A =_{\text{def}} \{s \mid s \text{ was a student at UCC last year}\}$
   $B =_{\text{def}} \{s \mid s \text{ is a student who left UCC at the end of last year}\}$
   $C =_{\text{def}} \{s \mid s \text{ is a newly arrived UCC student}\}$
   Write an expression that, if evaluated, would compute UCC's current students.
3. $A =_{\text{def}} \{s \mid s \text{ is a student who will take CS4000}\}$
   $B =_{\text{def}} \{s \mid s \text{ is a student who will take CS4409}\}$
   Why might the timetabling officer evaluate
   $$(A \cap B) = \emptyset$$

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Sets as members of sets: A set may have other sets as its members. E.g.

$$\{1, \{2,3\}, 3\}$$
$$\{\{1\}, \emptyset, \{1,3\}\}$$

Be extra careful with the distinction between $\in$ and $\subseteq$ now! E.g.

$$\{1\} \in \{\{1\}, \emptyset, \{1,3\}\}$$
$$1 \notin \{\{1\}, \emptyset, \{1,3\}\}$$
$$\{\{1\}\} \subseteq \{\{1\}, \emptyset, \{1,3\}\}$$

Exercise 4. $S1 =_{\text{def}} \{2,4,6\}$
$S2 =_{\text{def}} \{\emptyset, \{2\}, \{4\}, \{6\}, \{2,4\}, \{2,6\}, \{4,6\}, \{2,4,6\}\}$

As previously, rewrite each of the following expressions, replacing the dots $(\ldots)$ with $\in, \subseteq$ and/or $\subset$ to make as many true statements as possible.

1. $2 \ldots S1$
2. $2 \ldots S2$
3. $\{2\} \ldots S1$
4. $\{2\} \ldots S2$
5. $\{2,6\} \ldots S1$
6. $\{2,6\} \ldots S2$
7. $\{2,2,4,4,6,6\} \ldots S2$
8. $S1 \ldots S2$
9. $\emptyset \ldots S1$
10. $\emptyset \ldots S2$

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Let A be any non-empty set. A **partition** of A is a subdivision of all the members of A into disjoint, non-empty sets. More formally, a partition of A is a set of non-empty sets $\{A_1, A_2, \ldots, A_n\}$ such that

(i) for any $a \in A$, $a \in A_i$ for some i

(ii) for all i and j, $i \neq j$, $A_i \cap A_j = \emptyset$

Example: let $S =_{\text{def}} \{1,2,3,4,5,6,7,8,9\}$.

One partition is:

$$\{\{1,3,5\}, \{2,4,6,8\}, \{7,9\}\}$$

Other partitions include:

$$\{\{1,5,7\}, \{2,4,8\}, \{3,6,9\}\}$$
$$\{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}\}$$
$$\{\{1,2,3,4,5,6,7,8,9\}\}$$

The **powerset** of a set A is the set of all subsets of A. It is written $\wp(A)$ and defined as follows:

$$\wp(A) =_{\text{def}} \{a \mid a \subseteq A\}$$

E.g.

- $\wp(\{0,1\}) = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$.
- $\wp(\{a,b,c\}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$

Note how $\emptyset$ is always a member of the powerset. Why?

For any finite set A,

$$|\wp(A)| = 2^{|A|}$$

Exercise 5.

In this question, we are describing some second-year and fourth-year lecturers.

$$\begin{aligned} U &=_{\text{def}} \{DB, CS, HS, MvD, JH, KB, KH\} \\ Y2 &=_{\text{def}} \{KB, KH, HS, MvD, CS, DB\} \\ Y4 &=_{\text{def}} \{JH, KH, HS, DB\} \end{aligned}$$

1. Evaluate the following:
   (a) $Y2 \cup Y4$
   (b) $Y2 \cup Y2$
   (c) $Y2 \cap Y4$
   (d) $Y4 \cup U$
   (e) $Y4 \cup \emptyset$
   (f) $Y4 \cup \{\emptyset\}$
   (g) $Y2 \cap \emptyset$
   (h) $\{\emptyset\} \cap Y2$
   (i) $Y2 \cap U$
   (j) $Y2'$
   (k) $Y4''$
   (l) $U \setminus Y2$
   (m) $Y2 \setminus Y4$
   (n) $\wp(\{KB, CS\})$
   (o) $\wp(\{KB, CS, HS\})$
   (p) $\wp(\{KB\})$
   (q) $\wp(\emptyset)$
   (r) $\wp(\wp(\emptyset))$
2. Write down *all* the possible partitions of $Y4$. How many are there?

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A **couple** or **ordered pair** or **2-tuple** consists of *two* objects *taken in a certain order*. E.g.

$$\langle 1, 2 \rangle \qquad \langle 1, 1 \rangle$$

$$\langle DB, HS \rangle \qquad \langle DB, \{CS2205, CS4401, CS4409\} \rangle$$

$$\langle DB, \langle CS2205, \langle CS4401, CS4409 \rangle \rangle \rangle \qquad \langle DB, \langle \langle CS2205, CS4401 \rangle, CS4409 \rangle \rangle$$

Remember: order matters!

Cartesian product: For any two sets A_1 and A_2, the **Cartesian product** of A_1 and A_2, denoted by $A_1 \times A_2$, is defined as follows:

$$A_1 \times A_2 =_{\text{def}} \{\langle a_1, a_2 \rangle \mid a_1 \in A_1 \text{ and } a_2 \in A_2\}$$

Examples: Let $S1 =_{\text{def}} \{1,2,3\}$ and $S2 =_{\text{def}} \{a,b\}$.

$$\begin{aligned} S1 \times S2 &= \{\langle 1,a \rangle, \langle 2,a \rangle, \langle 3,a \rangle, \langle 1,b \rangle, \langle 2,b \rangle, \langle 3,b \rangle\} \\ S2 \times S1 &= \{\langle a,1 \rangle, \langle a,2 \rangle, \langle a,3 \rangle, \langle b,1 \rangle, \langle b,2 \rangle, \langle b,3 \rangle\} \\ S2 \times S2 &= \{\langle a,a \rangle, \langle a,b \rangle, \langle b,a \rangle, \langle b,b \rangle\} \end{aligned}$$

Exercise 6.

1. $S =_{\text{def}} \{a, b\}$. Evaluate $\wp(S) \times S$.
2. For finite sets A and B, what, in general, is the cardinality of $A \times B$?

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