Lecture 31: Reductions

Aims:

- To discuss the idea of a reduction and reducibility;
- To discuss the idea of polynomial-time reductions, and to see what we can learn from them;
- To see what we can learn from other reductions too.

31.1 Reductions

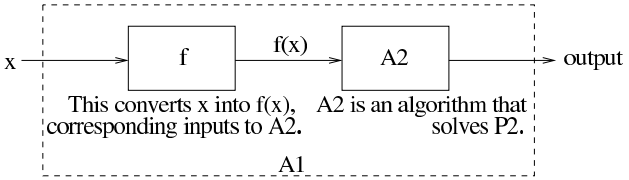
- *Reductions* of one problem to another are going to be very important in several upcoming lectures. The following joke might give you an intuition about what a reduction is:

  What's the difference between a mathematician and an engineer?
  Put an empty kettle in the middle of the kitchen floor and tell them to boil some water.
  The engineer will fill the kettle with water, put it on the stove, and turn the stove on. The mathematician will do the same thing.
  Next, put the kettle already filled with water on the stove, and ask them to boil some water.
  The engineer will turn the stove on.
  The mathematician will empty the kettle and put it in the middle of the kitchen floor... thereby reducing the problem to one that has already been solved!

- Suppose you have a problem P_2 which you know how to solve, e.g. by using algorithm A_2.
- Suppose you are given another problem P_1 that seems similar to P_2. How might you solve P_1?
  - You could try to solve P_1 from scratch.
  - You could try to borrow elements of A_2.
  - You could try to find a *reduction* from P_1 to P_2.
- A reduction of P_1 to P_2:
  - transforms inputs to P_1 into inputs to P_2;
  - runs A_2 (which solves P_2) as a 'black-box'; and
  - interprets the outputs from A_2 as answers to P_1.

  More formally,

A problem P_1 is *reducible* to a problem P_2 if there is a function f that takes any input x to P_1 and transforms it to an input $f(x)$ of P_2, such that the solution to P_2 on $f(x)$ is the solution to P_1 on x.



A1 is an algorithm that solves P1.

- Here's a simple example.

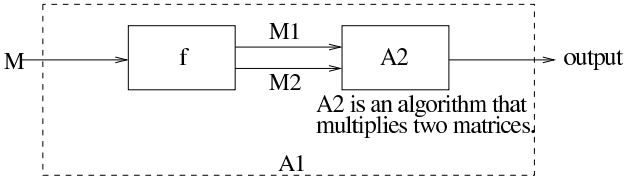
  Suppose we already have an algorithm for solving the following problem:

Problem 31.1. *Matrix Multiplication*
Parameters: *Two matrices, M_1 and M_2.*
Returns: *The result of multiplying M_1 and M_2 together.*

Suppose we are now asked to solve the following problem:

Problem 31.2. *Squaring a Matrix*
Parameters: *A matrix, M.*
Returns: *The result of squaring M.*

Here's how we would do it using a reduction:



A1 is an algorithm that squares a matrix.

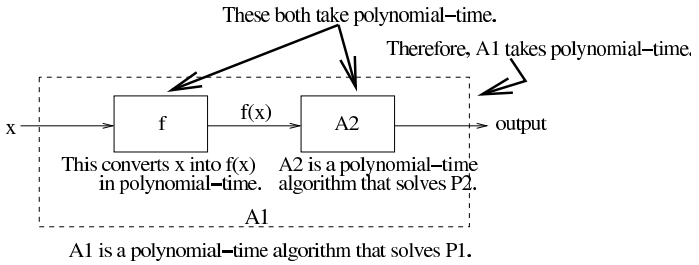
Class Exercise

What does f need to do?

- Our treatment of this topic so far makes it sound as if reductions are all about saving programming effort.
- But in theoretical Computer Science, reductions give us information. Under certain circumstances that we'll now investigate in detail, information we know about P_1 might apply to P_2, or information we know about P_2 might apply to P_1. Let's look at the details of this.

31.2 Polynomial-Time Reducibility

- The notion of reductions become especially useful in Complexity Theory when we can place a bound on the complexity of the transformation, f. Suppose, for example, that we know f takes polynomial time.
- We say that a problem P_1 is *polynomial-time reducible* to a problem P_2 if there is a function f, computable in polynomial time, that takes any input x to P_1 and transforms it to an input $f(x)$ of P_2, such that the solution to P_2 on $f(x)$ is the solution to P_1 on x.
- Shorthand $P_1 \xrightarrow{\text{poly}} P_2$
- Suppose $P_1 \xrightarrow{\text{poly}} P_2$.
  - This rules out the possibility that P_1 is intractable while P_2 is tractable.
  - Highly informally, it means that P_2 is 'as hard as' P_1.
  - It also means if P_2 is tractable, then P_1 is tractable.
  - Equivalently (the contrapositive): if P_1 is proved to be intractable, then P_2 is also intractable.
- Let's see why.
- **Theorem:** Suppose $P_1 \xrightarrow{\text{poly}} P_2$. If P_2 is tractable, then P_1 is tractable.
- *Proof:* If P_2 tractable, then we will have a polynomial-time algorithm A_2 for solving P_2. From this, we can construct a polynomial-time algorithm, A_1, for solving P_1, as follows:



The conversion, f, takes polynomial-time and A_2 takes polynomial-time, so A_1 (which comprises f followed by A_2 in sequence) must take polynomial-time.

Hence P_1 would be tractable too.

31.2.1 Proving a problem to be tractable

- Example 1
  - As an example, let's think about matrix multiplication and matrix squaring again.
  - We've already seen that Squaring a Matrix reduces to Matrix Multiplication.

    And, in fact, Squaring a Matrix $\xrightarrow{\text{poly}}$Matrix Multiplication.
  - Suppose Matrix Multiplication is tractable.

  - What do you conclude about Squaring a Matrix and why?
- Example 2.
  - Consider a decision problem P. (It returns YES or NO.)
  - The *complement* of P returns YES whenever P returns NO, and returns NO whenever P returns YES.
  - E.g.:

    Problem 31.3.
    Parameters: *An integer x and a finite-length list of integers, L.*
    Returns: *YES if x is a member of L and NO otherwise.*
  - Its complement is

    Problem 31.4.
    Parameters: *An integer x and a finite-length list of integers, L.*
    Returns: *NO if x is a member of L and YES otherwise.*
  - Suppose P_2 is a decision problem and that P_2 is tractable.
  - Suppose P_1 is the complement of P_2.
  - **Class Exercise:** Show that P_1 is tractable.

31.2.2 Proving a problem to be intractable

- Suppose $P_1 \xrightarrow{\text{poly}} P_2$.
- We have been using this:
  - If P_2 is tractable, then P_1 is tractable.
- But we can also use this (the contrapositive):
  - If P_1 is proved to be intractable, then P_2 is also intractable.

  Note which way round this is!!!!
- **Class Exercise:** Explain why this is so.
- In fact, any lower bound we prove for P_1 may apply to P_2.

31.3 Non-computability

- We have been using reductions to give us results in Complexity Theory. But we will see reductions again in the next part of this module, when we discuss computability.
- Reductions are a powerful tool for proving a problem to be non-computable.

- Suppose P_1 reduces to P_2. (Here we no longer require polynomial-time reducibility.)
- If P_1 is non-computable, then P_2 is also non-computable.
- **Class Exercise:** Explain why this is so.

Acknowledgements:

The idea of introducing reductions using the old joke about mathematicians and engineers comes from [Man89].

References

[Man89] U. Manber. *Introduction to Algorithms: A Creative Approach*. Addison Wesley, 1989.