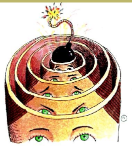

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Lecture 30: Graph Problems

Aims:

- To study graphs and graph problems;
- To use graph problems to exemplify tractable and intractable problems.

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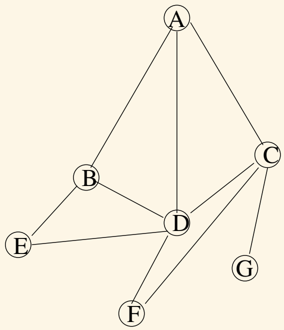
30.1. Graph Problems

30.1.1. Definition of a Graph

- The word ‘graph’ has two meanings in mathematics. The graph of a function, f, is all pairs $\langle x, f(x)\rangle$, coventionally plotted on a Cartesian plane (i.e. a piece of graph paper). (There’s a similar definition of the graph of a relation.) This is the meaning you are used to from school.
- But, there is another meaning of the word ‘graph’ that comes from the branch of maths known as *graph theory*, and it is this alternative meaning that we are exploring in the rest of this lecture.
- Here is an example of a graph:



- More formally, a graph G consists of two sets V and E,

$$G = \langle V, E\rangle$$

  – V is a finite, non-empty set of *vertices*;

  - E is a set of *edges*, each denoted by a pair of vertices.

  So the graph we showed earlier would be specified by the following two sets:

  $V = \{A, B, C, D, E, F, G\}$

  $E = \{\{A, B\}, \{A, C\}, \{A, D\}, \{B, D\}, \{B, E\}, \{C, D\}, \{C, F\}, \{C, G\}, \{D, E\}, \{D, F\}\}$

- In *undirected graphs*, edges have no direction. So an edge $\{u, v\}$ is the same as an edge $\{v, u\}$. These are the kinds of graphs we will be dealing with in this lecture.

- In *directed graphs*, edges have direction. Pictorially, you would draw an arrowhead on each line. So edge $\langle u, v\rangle$ is not the same as an edge $\langle v, u\rangle$. We are not dealing with these.

- I'm not going to say anything about data structures for storing graphs. Other modules should cover this material. Relevant key phrases are 'adjacency matrix' and 'adjacency list'.

- Here are some concepts from graph theory:

  Incident: Edge $\{u, v\}$ is said to be *incident* on vertices u and v. E.g. $\{A, B\}$ is incident on A and B.

  Degree: The *degree* of a vertex is the number of edges incident to that vertex. E.g. A has degree 3.

  Path: A *path* from vertex u to vertex z in graph G is a sequence of vertices $u, v, w, \ldots, y, z$ such that $\{u, v\}$, $\{v, w\}$,. . . , $\{y, z\}$ are edges in G. E.g. A, D, F, C is one path from A to C.

  Cycle: A *cycle* is a path in which the first and last vertices are the same. E.g. A, D, B, A is a cycle.

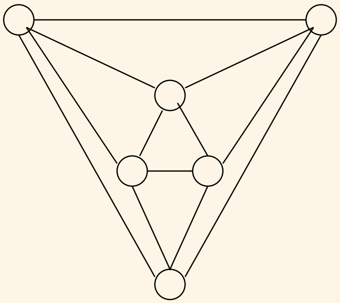
  Connected vertices: In a graph G, two *vertices* u and v are *connected* iff there is a path in G from u to v. E.g. A and C are connected.

Connected graph: A *graph* G is *connected* iff for every pair of distinct vertices u and v ($u \neq v$), there is a path from u to v. E.g. the graph shown earlier *is* connected.

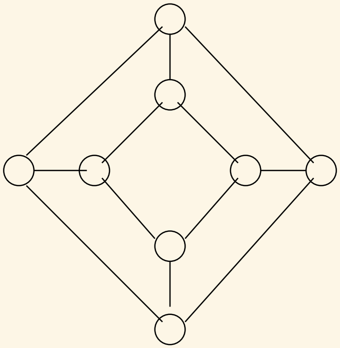
- Computer Scientists have developed numerous algorithms for processing graphs. There are algorithms for, e.g., finding paths, finding minimum cost paths, finding cycles, finding minimum cost cycles, constructing spanning trees especially minimum cost spanning trees, and so on and so on.
- This might all sound somewhat abstract. But graphs and their algorithms have many very real applications. Graphs can represent road networks, railway networks, computer networks, the layout of components and wires in a circuit, and so on.

30.2. Eulerian Cycles

- A *Eulerian cycle* is a cycle that includes every *edge* exactly once.
- Here we show a graph that has a Eulerian cycle:



- And here's a graph that has no Eulerian cycle:



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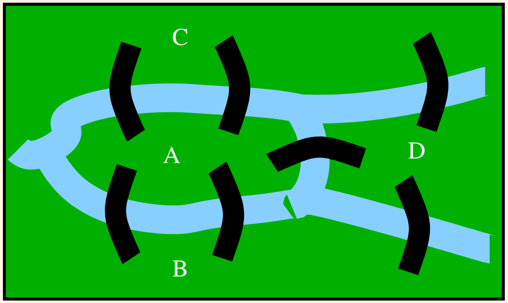
- Here are two related problems that concern Eulerian cycles and that are of interest to Computer Scientists:

Problem 30.1. *Eulerian Cycle Decision Problem*
Parameters: *A graph* $G = \langle V, E \rangle$*.*
Returns: *YES if there is an Eulerian cycle; NO otherwise.*

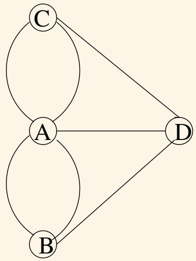
Problem 30.2. *Eulerian Cycle Search Problem*
Parameters: *A graph* $G = \langle V, E \rangle$*.*
Returns: *A Eulerian cycle in* G*, if there is one.*

- Interest in Eulerian cycles has its origins in a puzzle. In the town of Königsberg (now Kalingrad), the river Pregel is crossed by seven bridges, as follows:



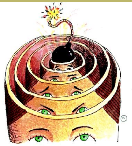
Is it possible for the citizens of Königsberg to take a stroll from some starting point, to cross each bridge exactly once and return to their starting point?

- Here's the town shown in a more abstract way:



- Leonhard Euler, a Swiss mathematican, solved this problem in 1736. Euler proved that it is *not* possible for the citizens of Königsberg to complete such a stroll. In fact, Euler solved the more general problem that we gave earlier: for any graph, he worked out how we can determine whether there is such a cycle.
- There is a Eulerian cycle if and only if
  - the graph is connected, and
  - the degree of every vertex is even.
- Why? Informally, you must leave each vertex via an edge that is different from the edge that took you to that vertex.

  This is obviously a necessary condition for there to be a Eulerian cycle. But Euler proved that it is both necessary and sufficient.
- Let's consider the problem complexity of the two problems that we gave earlier.



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- Eulerian Cycle Decision Problem
  - Algorithm:
    - Check the graph is connected. (There is a polynomial-time algorithm to do this.)
    - Check each vertex has even degree. (This is also polynomial.)
  - Therefore, this is a problem with a known polynomial-time algorithm.
- Eulerian Cycle Search Problem
  - Algorithm, e.g. Fleury's algorithm (polynomial)

```
Check that a Eulerian cycle can exist (see above);
v := choose any vertex;
while there are edges
{    Choose an edge, e, that is incident on v but do not
                              choose a bridge unless you have to;
     v := the other vertex that is incident on e;
     Insert e into the answer;
     Delete e from G;
}
return the answer;
```

(A bridge is an edge whose deletion would cause the graph to no longer be a connected graph.)

  - Therefore, this is a problem with a known polynomial-time algorithm.

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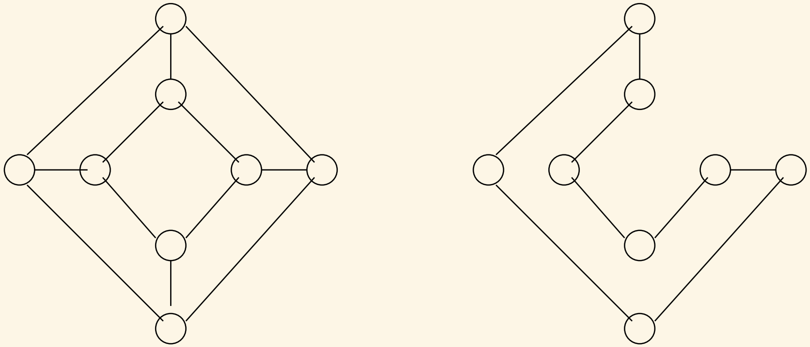
30.3. Hamiltonian Cycles

- A *Hamiltonian cycle* is a cycle that contains every vertex exactly once (except that the initial and final vertices will be equal).
- Here we show a graph and, on the right, we highlight a Hamiltonian cycle through that graph.



- Here are two related problems that concern Hamiltonian cycles and that are of interest to Computer Scientists:

Problem 30.3. *Hamiltonian Cycle Decision Problem*

Parameters: *A graph* $G = \langle V, E \rangle$.

Returns: *YES if there is a Hamiltonian cycle; NO otherwise.*

Problem 30.4. *Hamiltonian Cycle Search Problem*

Parameters: *A graph* $G = \langle V, E \rangle$.

Returns: *A Hamiltonian cycle in* G*, if there is one.*

- Let's consider the problem complexity of the two problems that we gave earlier.
  - Hamiltonian Cycle Decision Problem and Search Problem
    * The only known algorithms are exponential.
    * If n is the number of vertices,
      · E.g. generate-and-test all permutations of the vertices: $O(n^n)$
      · E.g. use a backtracking algorithm: $O(n!)$
      · There may be better algorithms, but they're all exponential.
      · But the problem has not been proven to be intractable.

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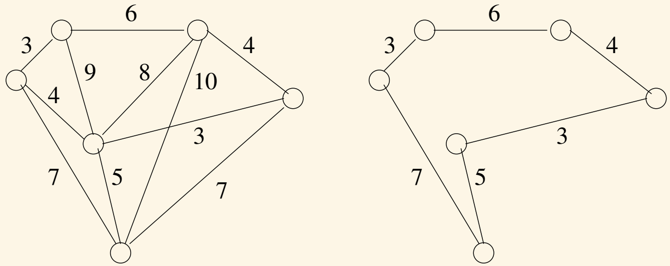
30.4. Traveling Salesperson Problem (TSP)

- The TSP applies to *weighted graphs* (ones with costs on the edges). The TSP is about finding minimum cost Hamiltonian cycles.

- Here we show a graph and, on the right, we highlight a minimum cost Hamiltonian cycle:



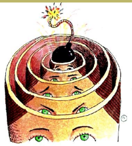
(Not to scale)

- Here are two related problems that concern TSPs and that are of interest to Computer Scientists:

Problem 30.5. *TSP Decision Problem*

Parameters: *A weighted graph* $G = \langle V, E \rangle$ *and an integer* k*.*

Returns: *YES if there is a Hamiltonian cycle whose total cost is less than or equal to* k*; NO otherwise.*



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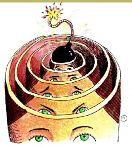
Problem 30.6. *TSP Search Problem*
Parameters: *A weighted graph* $G = \langle V, E \rangle$.
Returns: *A minimum cost Hamiltonian cycle.*

- The problem complexity story here is the same as that for the Hamiltonian cycle problems:
  - TSP Decision Problem: no known polynomial-time algorithm but not yet proven to be intractable.
  - TSP Search Problem: no known polynomial-time algorithm but not yet proven to be intractable.



30.5. Summary

- Despite the strong similarities between the problems we have looked at in this lecture, they're complexities are quite different:
  - Problems with known polynomial-time algorithms:
    - Eulerian Cycle Decision Problem and Search Problem.
  - Problems that have been proven to be intractable:

    Problem 30.7. *Hamiltonian Cycle Enumeration Problem*
    Parameters: *A graph* $G = \langle V, E \rangle$.
    Returns: *All Hamiltonian cycles in* G.

    In the worst case, there can be an exponential number of Hamiltonian cycles. Therefore, this is a problem that is provably intractable.
  - Problems with no known polynomial-time algorithm but not yet proven to be intractable:
    - Hamiltionian Cycle Decision Problem and Search Problem.
    - TSP Decision Problem and Search Problem.

Acknowledgements:

This material is partly based on topics in [Wik].

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References

[Wik] Wikipedia: The free encyclopedia. http://en2.wikipedia.org/.