# Lecture 21: Total Correctness

Aims:

- To look at the inference rules for total correctness.

## 21.1 Total Correctness

- Suppose we are asked to prove

  $\vdash_{\text{tot}} (\!| P |\!)\, C\, (\!| Q |\!)$

  So now we are concerned with termination as well as partial correctness.

- We need a different deduction system with a different set of inference rules. In a language as simple as MCCCA, there are only two kinds of problem that can prevent termination.
  - The value of an expression may be undefined, e.g. division by zero.
  - A **while** loop may never be exited.

  Of course, if we extend our language, then there could be other sources of non-termination. For example, if we add procedures and we allow recursion then any command that contains a call to a recursive procedure (including cases of mutual recursion) may result in an infinite computation.

## 21.2 A Deduction System for Total Correctness

- Most of the inference rules are unchanged. But wherever we evaluate an expression, we must make sure that its value is defined. For example, if we have the following assignments,

  $$x := 10 \operatorname{div} u$$

  $$x := 32 \bmod v$$

  $$x := z \operatorname{div} (w + 1)$$

  $$x := 1 + z \bmod (x - y)$$

  we must make sure that $u$, $v$, $(w+1)$ and $(x-y)$ are not zero, respectively.

  If we have an **if** command or a **while** loop, the Boolean expression may involve some arithmetic, and again this gives the potential for a division by zero. Here are examples:

  $$\textbf{if } x = 10 \operatorname{div} y \ldots$$

  $$\textbf{while } x = 32 \bmod y \ldots$$



- In our new inference rules, we write '$E$ is defined'. But you don't write this in a proof. You replace it by a suitable assertion. So if, within $E$, there's a division (using div or mod) by some expression $E'$, you replace '$E$ is defined' by $E' \neq 0$. If there's no division, then simply ignore '$E$ is defined'.

  Sequencing

  $$\frac{(\!| P |\!)\, C_1\, (\!| R |\!), \quad (\!| R |\!)\, C_2\, (\!| Q |\!)}{(\!| P |\!)\, C_1; C_2\, (\!| Q |\!)}$$

  Assignment

  $$\frac{E \text{ is defined}}{(\!| Q[V \mapsto E] |\!)\, V := E\, (\!| Q |\!)}$$

  Consequence

  $$\frac{P \Rightarrow P', \quad (\!| P' |\!)\, C\, (\!| Q' |\!), \quad Q' \Rightarrow Q}{(\!| P |\!)\, C\, (\!| Q |\!)}$$

  One-armed-conditional

  $$\frac{B \text{ is defined}, \quad (\!| B \wedge P |\!)\, C\, (\!| Q |\!), \quad (\neg B \wedge P) \Rightarrow Q}{(\!| P |\!)\, \textbf{if } B\, C\, (\!| Q |\!)}$$

  Two-armed-conditional

  $$\frac{B \text{ is defined}, \quad (\!| B \wedge P |\!)\, C_1\, (\!| Q |\!), \quad (\!| \neg B \wedge P |\!)\, C_2\, (\!| Q |\!)}{(\!| P |\!)\, \textbf{if } B\, C_1 \textbf{ else } C_2\, (\!| Q |\!)}$$

- The rule for **while** loops, however, is much more complicated.

  While

  $$\frac{\begin{array}{c} B \text{ is defined}, \\ (\!| Inv \wedge B \wedge 0 \leq VE \wedge VE = VE_0 |\!)\, C\, (\!| Inv \wedge 0 \leq VE \wedge VE < VE_0 |\!) \end{array}}{(\!| Inv \wedge 0 \leq VE |\!)\, \textbf{while } B\, C\, (\!| Inv \wedge \neg B |\!)}$$

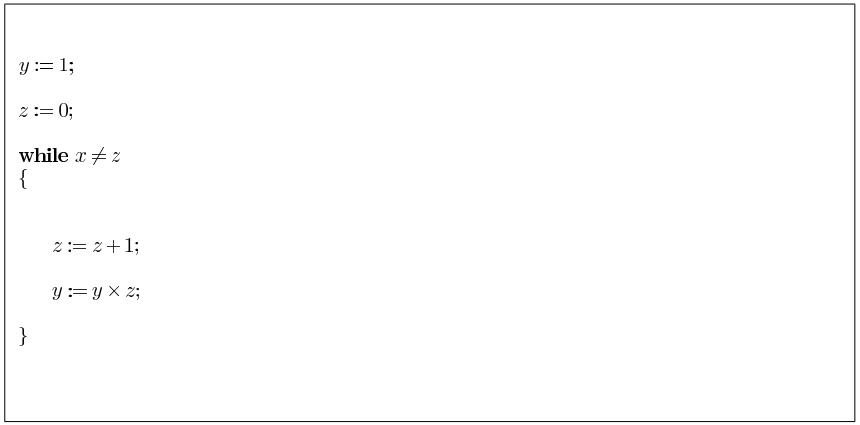
- We have to try to discover a *variant*. A *variant* is an integer expression whose value can be shown to decrease every time we go round the loop, but which is always non-negative. If we can find an expression that has these properties, the loop must terminate. The expression can only decrease in value a finite number of times before it becomes zero.

  In the rule above $VE$ is the variant. We require that its value decreases by saying in the condition of the rule that, if its value is $VE_0$ before the body, then its value is less than $VE_0$ after the body.

- How do we prove the total correctness of a **while** loop?

  1. Guess a wff $Inv$ that you hope is an invariant and an integer expression $VE$ that you hope is a variant.
  2. Prove that $(Inv \wedge \neg B) \Rightarrow Q$.
  3. Push $Inv \wedge 0 \leq VE \wedge VE < VE_0$ upwards through $C$. Let's call the wff you get from this $W$.
  4. Prove that $(Inv \wedge B \wedge 0 \leq VE \wedge VE = VE_0) \Rightarrow W$.
  5. Now write $Inv \wedge 0 \leq VE$ above the **while** loop. (Continue to push this up through the rest of the program, if any.)

- Prove that $\vdash_{\text{tot}} (\!| x \geq 0 |\!)\ ProgA\ (\!| y = x! |\!)$ where $ProgA$ is:




  - The invariant is $y = z!$ as before.
  - The variant is $x - z$.

  We can see that loop-test $B$, i.e. $x \neq z$, and all the expressions in the assignment commands are defined (no uses of div or mod).

**Acknowledgements**

My approach here was partly based on the approach taken in [Kal90] as well as [HR00].

## References

[HR00] M. Huth and M. Ryan. *Logic in Computer Science: Modelling and Reasoning about Systems*. Cambridge University Press, 2000.

[Kal90] A. Kaldewaij. *Programming: The Derivation of Algorithms*. Prentice Hall, 1990.