Lecture 15:
Proof

Aims:

- To see more examples of natural deduction;
- To see examples of formal proofs;
- To discuss what we mean by the soundness and completeness of a deduction system;
- To see examples of informal proofs.
15.1. More Derivations

- We start with three more examples of what we were doing at the end of the previous lecture.
  - Show \( \{ p \leftrightarrow q, q \Rightarrow \neg r \} \vdash p \Rightarrow \neg r \)
  - Show \( \{ p \Rightarrow r, q \Rightarrow r, p \lor q \} \vdash r \)
  - Show \( \{ p \Rightarrow q, \neg q \} \vdash \neg p \)
15.2. Formal Proof

- A proof is a derivation from no premisses. Obviously, they’re almost bound to begin with some assumptions!

- E.g. prove that $\vdash p \Rightarrow p$
  1. Subderivation:
   1.1 $p$ assumption
   1.2 $p$ 1.1, repetition
  2. $p \Rightarrow p$ 1, $\Rightarrow$I

- Here’s another example of a proof. We’ll prove that $\vdash ((p \land q) \lor (\neg p \land r)) \Rightarrow (q \lor r)$
  1.
    1.1
    1.2
      1.2.1
      1.2.2
      1.2.3
    1.3
      1.3.1
      1.3.2
      1.3.3
    1.4
  2.
15.3. Soundness and Completeness

- A deduction system derives wffs using syntactic operations alone, without reference to the semantics. We could invent a set of inference rules that license even quite bizarre inferences. However, if a deduction system is to be useful, the wffs we can derive from a set of wffs must tie in with logical consequences of that set.

At least two things may go wrong.

- An inference rule might license an inference that is not a logical consequence. For example,
\[
\Rightarrow\text{-DUFF}
\]
\[
\frac{W_1 \Rightarrow W_2, W_2}{W_1}
\]

We can use this rule to show, e.g., \(\{p \Rightarrow q, q\} \vdash p\). But, we know that \(\{p \Rightarrow q, q\} \nvDash p\). (We showed this earlier. If you can’t find it, draw a truth table and confirm it again!)

- An inference rule of this kind is said to be unsound. A deduction system that contains such a rule is unsound.

An inference rule is sound if the conclusions one can infer from any set of wffs using the rule are logical consequences of the set of wffs.

A deduction system is sound if it contains only sound inference rules.

- Another thing that may go wrong is that there may be logical consequences of a set of wffs that the deduction system fails to derive.

For example, we know that \(\{p \lor q, \neg p\} \models q\). (Confirm it with a truth table if you don’t believe it!). But if our deduction system has insufficient inference rules or insufficiently ‘powerful’ inference rules, it may be that we cannot show that \(\{p \lor q, \neg p\} \vdash q\). In this case, the deduction system is said to be incomplete.
A deduction system is complete if every logical consequence of any set of wffs can also be derived from the set of wffs.

• Here’s a summary of soundness and completeness.
  A deduction system is sound so long as for any set of wffs $A$ and any wff $W$,
  
  $$\text{if } A \vdash W \text{ then } A \models W$$

  A deduction system is complete so long as for any set of wffs $A$ and any wff $W$,
  
  $$\text{if } A \models W \text{ then } A \vdash W$$

• Soundness is essential, but completeness might be sacrificed in Computer Science uses of deduction systems, where the efficiency of the automated deduction system is important.
  
The soundness and completeness of a deduction system is obviously something that the logician who proposes the system should prove.

• If a deduction system is sound and complete, then for any wff $W$,
  
  $W$ is valid if and only if $\vdash W$

  i.e. any valid wff will be provable and vice versa.
  And, for any set of wffs $A$ and wff $W$,
  
  there will be no interpretation that satisfies all the wffs in $A$
  
  if and only if $A \vdash W \land \neg W$

  (When a wff of the form $W \land \neg W$ can be derived from a set of wffs $A$, we say that $A$ is inconsistent.)
15.4. Informal Proof

- In the correctness proofs that we will be doing soon, we won’t be working with propositional symbols ($p$s and $q$s). We will have statements about programs, e.g. statements about the values of the variables in the programs, such as $x > 2$ or $x = 0 \land y > 10$.

  Mostly, we’ll need to prove that a conditional holds.

- Suppose we need to prove, e.g., $\vdash x > 5 \Rightarrow x > 0$.

  Strictly, in propositional logic this cannot be proved. Think of it written using $p$s and $q$s! There are two different statements, linked by a conditional. So you are being asked to prove $\vdash p \Rightarrow q$, which is simply not provable.

  But you can’t help inspecting the inner details of the two statements and realising, *given additionally what you know about arithmetic*, that it is provable.

- We will prove such statements in two (informal) ways. (It is up to you to be smart enough to choose which of the two to use in any particular case.)

- **Method 1.** Use the laws of the algebra of propositions, plus whatever you know about arithmetic, to show the statement is valid (i.e. always true).

  **Example.** To show $\vdash \text{True} \Rightarrow x + x = 2x$.

- **Method 2.** Use the rules of natural deduction, plus whatever you know about arithmetic.

  **Example.** To show $\vdash x = n \Rightarrow x + 1 = n + 1$.

Acknowledgements

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