MATHS FOR INFORMATION PROCESSING

Problem Sheet 4

1

(i) Say which of the following expressions are *syntactically well-formed formulae* of propositional logic and which are not.

(a) $((p_1 \wedge p_2) \vee \neg p_2)$

(b) $(p_1 \vee p_2 \vee p_3)$

(c) $\neg(p_1 \wedge p_4) \Leftrightarrow p_6$

(d) $(\neg p_1 \wedge p_2)$

(e) $(p_1 \Rightarrow (\neg(p_2 \vee p_1)))$

(f) $\neg\neg\neg\neg p_3$

(g) $\neg(\neg p_1 \wedge \neg p_8)$

(h) $((p_1 \vee p_2) \vee p_3)$

(i) $(p_1 \Rightarrow p_2 \vee p_3)$

(j) $p_1 \vee p_2 \Rightarrow p_3 \wedge p_4$

(k) $p_1 \Rightarrow p_2 \Rightarrow p_3$

(l) $p_1 \vee p_2 \wedge p_3$

(ii) For those expressions from above that you said were well-formed, rewrite the expressions, inserting additional parentheses to make explicit the way that the operator precedence and associativity conventions given in your notes would affect the evaluation order.

2

In this question,

$$
\begin{aligned}
p_1 &\triangleq \text{Clyde is going to the beach} \\
p_2 &\triangleq \text{Clyde is going to the cinema} \\
q_1 &\triangleq \text{Flopsy is going to the beach} \\
q_2 &\triangleq \text{Flopsy is going to the cinema}
\end{aligned}
$$

Write each of the following English statements in symbolic form, i.e. as wffs of propositional logic.

In those cases where you believe that the full meaning is not captured, give the best translation into symbolic form that you can, but add a note in English to your answer that records what aspect of the meaning you think is not being adequately captured.

In those cases where you believe that the English is ambiguous, give as many translations as there are meanings.

(i) Clyde is going to the beach but Flopsy is going to the cinema.

(ii) Clyde is going to the beach or the cinema, and so is Flopsy.

(iii) If Clyde goes to the beach while Flopsy does not, Flopsy will go to the cinema.

(iv) Clyde is going to the beach, and if Clyde is going to the beach, so is Flopsy.

(v) It is not the case that Clyde is going to the beach and Flopsy is not.

(vi) Clyde is not only going to the beach but to the cinema too.

(vii) If Clyde isn't going to the beach, then Clyde and Flopsy are going to the cinema, whether Flopsy goes to the beach or not.

(viii) If Clyde goes to the beach if Flopsy goes to the beach, then Clyde is going to the beach.

(ix) Clyde goes to the beach if Flopsy does, and the other way round.

(x) Clyde is going to the beach if Flopsy doesn't.

(xi) Clyde goes to the beach only if Flopsy doesn't.

(xii) Clyde goes to the beach exactly if Flopsy doesn't.

(xiii) Clyde goes to the beach when Flopsy doesn't.

(xiv) Clyde goes to the beach provided Flopsy doesn't.

(xv) Clyde goes to the beach unless Flopsy does.

3

Draw truth tables for the following wffs *and hence* state whether each is valid or invalid, and satisfiable or unsatisfiable.

(i) $(p \Rightarrow q) \Rightarrow (p \wedge q)$

(ii) $(p \Rightarrow q) \vee \neg(p \Leftrightarrow \neg q)$

(iii) $p \wedge (q \Leftrightarrow \neg q)$

(iv) $p \vee \neg(p \wedge q)$

(v) $p \vee \neg(p \wedge p)$

(vi) $\neg(\neg(p \wedge q) \wedge \neg(p \vee q))$

(vii) $((p \wedge \neg q) \vee (\neg p \wedge q)) \Rightarrow r$

4

Here are some valid wffs:

$$p \Rightarrow p$$

$$p \vee \neg p$$

$$\neg(p \wedge \neg p)$$

$$((p_1 \Rightarrow p_2) \wedge (p_2 \Rightarrow p_3)) \Rightarrow (p_1 \Rightarrow p_3)$$

Here's an unsatisfiable wff:

$$p \wedge \neg p$$

(You can optionally confirm that these wffs are valid/unsatisfiable by drawing truth-tables.This would be good practice for you.)

(i) If a wff is valid, then its negation is unsatisfiable. Why?

(ii) If a wff is unsatisfiable, then its negation is valid. Why?

(iii) *Hence*, answer the following *without drawing truth-tables*.

 (a) Is $\neg(p \vee \neg p)$ valid or invalid; is it satsifiable or unsatisfiable?

 (b) Is $\neg(p \wedge \neg p)$ valid or invalid; is it satisfiable or unsatisfiable?

(iv) Any valid wff can be turned into a template for valid wffs. For example, from the valid wffs given earlier, we can get the following:

For any wffs W, W_1, W_2 and W_3,

$$W \Rightarrow W \text{ is valid}$$

$$W \vee \neg W \text{ is valid}$$

$$\neg(W \wedge \neg W) \text{ is valid}$$

$$((W_1 \Rightarrow W_2) \wedge (W_2 \Rightarrow W_3)) \Rightarrow (W_1 \Rightarrow W_3) \text{ is valid}$$

In other words, no matter what wffs you substitute for W, W_1, W_2 and W_3 in these templates, the resulting wff will be a valid wff, e.g. substituting $(q \Leftrightarrow \neg q)$ for W in $W \vee \neg W$ gives the valid wff $(q \Leftrightarrow \neg q) \vee \neg(q \Leftrightarrow \neg q)$.

Confirm that this wff is valid by drawing its truth table.

(v) Making use of the information given in the earlier parts of this question, use the simplest method you can think of to determine whether the following wffs are valid or invalid, and satisfiable or unsatisfiable. (Make clear what your method is in each case.)

(vi) $(p \wedge q) \Rightarrow (p \wedge q)$

(vii) $\neg p \Rightarrow (\neg p \vee q)$

(viii) $\neg\neg((p_1 \vee p_2) \wedge \neg(p_1 \vee p_2))$

(ix) $\neg(\neg(p_1 \Leftrightarrow p_2) \Rightarrow (\neg(p_1 \Leftrightarrow p_2) \vee (p_3 \wedge (p_2 \vee p_1))))$

Use truth tables, plus a sentence or two of explanation, to show:

(i) $\{p \Leftrightarrow q, q\} \models p$

(ii) $\{p \Rightarrow \neg q, \neg r \Rightarrow p, q\} \models r$

(iii) $\{p \Leftrightarrow \neg q\} \not\models p \Rightarrow q$

(iv) $\{(q \vee p) \Rightarrow r\} \models q \Rightarrow r$

6

Without drawing truth tables, state whether the following hold. Include some words of explanation in your answer.

(i) $\{(p \vee q) \Rightarrow (q \wedge \neg r)\} \models (p \vee q) \Rightarrow (q \wedge \neg r)$

(ii) $\{\neg((p \vee q) \Rightarrow (q \wedge \neg r))\} \models (p \vee q) \Rightarrow (q \wedge \neg r)$

(iii) $\{(p \vee q) \Rightarrow (q \wedge \neg r), p \vee r\} \models p \vee \neg p$

(iv) $\{p, \neg p\} \models (p \vee q) \Rightarrow (q \wedge \neg r)$

7

(i) Use truth-tables to show that the following equivalences hold. Include enough columns to show your working.

(a) $\neg(W_1 \Rightarrow W_2) \equiv W_1 \wedge \neg W_2$

(b) $W_1 \Rightarrow W_2 \equiv \neg W_2 \Rightarrow \neg W_1$ (i.e. the contrapositive law)

(c) $(W_1 \wedge W_2) \Rightarrow W_3 \equiv W_1 \Rightarrow (W_2 \Rightarrow W_3)$

(ii) Repeat the above. This time show that the equivalences hold algebraically using the laws given in your 'Examination Notes'. (For (b), your manipulation may not use the contrapositive law!)

How many possible connectives are there?

So far, we have used five connectives ($\neg, \wedge, \vee, \Rightarrow$ and $\Leftrightarrow$). It is conventional to introduce these five to students of logic. The main reasons for choosing these five appears to be that these are the ones with moderately direct English paraphrases.

In the table below, I show these five and all the other possible binary connectives that we could have used. (You have to turn the table on its side for it to look like the tables we have used so far in this course.)

F	F	T	T	W_1
F	T	F	T	W_2
T	T	T	T	t
F	T	T	T	$W_1 \vee W_2$
T	F	T	T	$W_1 \Leftarrow W_2$
F	F	T	T	W_1
T	T	F	T	$W_1 \Rightarrow W_2$
F	T	F	T	W_2
T	F	F	T	$W_1 \Leftrightarrow W_2$
F	F	F	T	$W_1 \wedge W_2$
T	T	T	F	$W_1 \Uparrow W_2$
F	T	T	F	$W_1 \oplus W_2$
T	F	T	F	$\neg W_2$
F	F	T	F	$W_1 \not\Rightarrow W_2$
T	T	F	F	$\neg W_1$
F	T	F	F	$W_1 \not\Leftarrow W_2$
T	F	F	F	$W_1 \Downarrow W_2$
F	F	F	F	f

How many connectives do we need?

We can express all of the news one in terms of the original five.

Reverse conditional	$W_1 \Leftarrow W_2$	$\equiv$	$W_2 \Rightarrow W_1$
Nand	$W_1 \Uparrow W_2$	$\equiv$	$\neg(W_1 \wedge W_2)$
Nor	$W_1 \Downarrow W_2$	$\equiv$	$\neg(W_1 \vee W_2)$
Exclusive-or	$W_1 \oplus W_2$	$\equiv$	$\neg(W_1 \Leftrightarrow W_2)$
Inhibition	$W_1 \not\Rightarrow W_2$	$\equiv$	$\neg(W_1 \Rightarrow W_2)$
Reverse inhibition	$W_1 \not\Leftarrow W_2$	$\equiv$	$\neg(W_2 \Rightarrow W_1)$

Lots of other equivalences are also possible instead of the ones given above, e.g.

$$W_1 \oplus W_2 \equiv (W_1 \vee W_2) \wedge \neg(W_1 \wedge W_2)$$

Even t and f can be expressed using the original five:

$$t \equiv W \vee \neg W$$

$$f \equiv W \wedge \neg W$$

Can we manage with fewer than five?

We already know that we don't need $\Leftrightarrow$ (because $W_1 \Leftrightarrow W_2 \equiv (W_1 \Rightarrow W_2) \wedge (W_2 \Rightarrow W_1)$), and we don't need $\Rightarrow$ (because $W_1 \Rightarrow W_2 \equiv \neg W_1 \vee W_2$). So we're down to three: $\neg$, $\wedge$ and $\vee$.

(i) Show that we only need $\neg$ and $\wedge$ by expressing $\vee$ in terms of these two.

(ii) Show that, alternatively, we only need $\neg$ and $\vee$ by expressing $\wedge$ in terms of these two.

(iii) Show that, alternatively, we only need $\neg$ and $\Rightarrow$ by expressing $\wedge$ and $\vee$ in terms of these two.

(iv) In fact, we only need one connective: nand ($\Uparrow$) on its own will suffice. How do I know? Well, because we know from above that $\neg$ and $\wedge$ on their own are sufficient and I can express these two in terms of $\Uparrow$:

$$\neg W \equiv W \Uparrow W$$

$$W_1 \wedge W_2 \equiv (W_1 \Uparrow W_2) \Uparrow (W_1 \Uparrow W_2)$$

(Check these with truth tables.)

Similarly, show that just nor ($\Downarrow$) on its own will suffice.

There is a connection between propositional logic and the design of computer circuits, which you will see in your GNO course. The attempt to get a minimal set of connectives is useful to allow the manufacture of the minimal number of types of components. Equally, algebraic simplification in general can reduce the number of components needed in any one particular circuit.

9

(i) Show the following derivations hold. You will need only the rules $\wedge$-I, $\wedge$-E-L, $\wedge$-E-R, $\Rightarrow$-E, $\vee$-I-R, $\vee$-I-L, $\Leftrightarrow$-E=L, $\Leftrightarrow$-E-R and $\neg$-E.

(a) $\{p_1 \wedge p_2, p_3 \wedge p_4\} \vdash p_1 \wedge p_4$

(b) $\{p, p \Rightarrow q, q \Leftrightarrow r\} \vdash q \wedge r$

(c) $\{\neg\neg q\} \vdash q \vee r$

(d) $\{p_1 \wedge p_2, p_1 \Rightarrow p_3, p_2 \Rightarrow p_4\} \vdash p_3 \wedge p_4$

(ii) Show the following derivations hold. You will need only the rules mentioned above plus $\Rightarrow$-I and REPETITION.

(a) $\{(p \wedge q) \Rightarrow r, p\} \vdash q \Rightarrow r$

(b) $\{p\} \vdash q \Rightarrow p$

(iii) Prove the following. (You will need only the rules mentioned so far in this question.)

(a) $\vdash r \Rightarrow (p \Rightarrow (q \Rightarrow r))$

(b) $\vdash (q \Rightarrow r) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$

(iv) Show whether this inference rule is sound or not:

$$\frac{(W_1 \wedge W_2) \Rightarrow W_3}{W_1 \Rightarrow W_3}$$

Model Answers

Maths for Information Processing

Problem Sheet 4

Question 1

Part (i)
Option (a)
Yes
Option (b)
Yes
Option (c)
Yes
Option (d)
Yes
Option (e)
Yes
Option (f)
Yes
Option (g)
Yes
Option (h)
Yes
Option (i)
Yes
Option (j)
Yes
Option (k)
Yes
Option (l)
Yes

They are *all* well-formed formulae of propositional logic.

Part (ii)

(a) $((p_1 \wedge p_2) \vee (\neg p_2))$

(b) $((p_1 \vee p_2) \vee p_3)$

(c) $(\neg(p_1 \wedge p_4)) \Leftrightarrow p_6$

(d) $((\neg p_1) \wedge p_2)$

(e) $(p_1 \Rightarrow (\neg(p_2 \vee p_1)))$

(f) $\neg(\neg(\neg(\neg p_3)))$

(g) $\neg((\neg p_1) \wedge (\neg p_8))$

(h) $((p_1 \vee p_2) \vee p_3)$

(i) $(p_1 \Rightarrow (p_2 \vee p_3))$

(j) $(p_1 \vee p_2) \Rightarrow (p_3 \wedge p_4)$

(k) $p_1 \Rightarrow (p_2 \Rightarrow p_3)$

(l) $(p_1 \vee p_2) \wedge p_3$

Question 2

Part (i)

$p_1 \wedge q_2$: this loses the contrast conveyed by 'but'

Part (ii)

$(p_1 \vee p_2) \wedge (q_1 \vee q_2)$

Part (iii)

$(p_1 \wedge \neg q_1) \Rightarrow q_2$: a loss of 'contrast'

Part (iv)

$p_1 \wedge (p_1 \Rightarrow q_1)$

Part (v)

$\neg(p_1 \wedge \neg q_1)$. Some of you might believe this is ambiguous, having the following translation too: $\neg p_1 \wedge \neg q_1$. *Personally*, I don't think this reading is available, but there are no marks off for saying that you think it is available.

Part (vi)

$p_1 \wedge p_2$. Don't be fooled by the appearance of the word 'not' in this one. The 'not' isn't negating the sentence, it is negating the 'only'. Think about this sentence: it says Clyde is going to the beach, doesn't it? There is some loss of meaning, though.

Part (vii)

$\neg p_1 \Rightarrow ((q_1 \vee \neg q_1) \Rightarrow (p_2 \wedge q_2))$. There might be an ambiguity; the alternative reading would be $(q_1 \vee \neg q_1) \Rightarrow (\neg p_1 \Rightarrow (p_2 \wedge q_2))$

Part (viii)

$(q_1 \Rightarrow p_1) \Rightarrow p_1$

Part (ix)

$p_1 \Leftrightarrow q_1$

Part (x)

$\neg q_1 \Rightarrow p_1$

Part (xi)

$p_1 \Rightarrow \neg q_1$

Part (xii)

$p_1 \Leftrightarrow \neg q_1$

Part (xiii)

$\neg q_1 \Rightarrow p_1$: this assumes that 'when' is being used in a conditional-like manner; any temporal relationship is not captured with this translation.

Part (xiv)

$p_1 \Leftrightarrow \neg q_1$

Part (xv)

We need exclusive-or ($\oplus$) but this wasn't part of our syntax for propositional logic so we must use one of its equivalents: either $\neg(p_1 \Leftrightarrow q_1)$ or $(p_1 \vee q_1) \wedge \neg(p_1 \wedge q_1)$ will do.

Question 3

Part (i)

p	q	$(p$	$\Rightarrow$	$q)$	$\Rightarrow$	$(p$	$\wedge$	$q)$
T	T	T	T	T	T	T	T	T
T	F	T	F	F	T	T	F	F
F	T	F	T	T	F	F	F	T
F	F	F	T	F	F	F	F	F
					$\triangle$			

Invalid, satisfiable.

Part (ii)

p	q	$(p$	$\Rightarrow$	$q)$	$\vee$	$\neg$	$(p$	$\Leftrightarrow$	$\neg$	$q)$
T	T	T	T	T	T	T	T	F	F	T
T	F	T	F	F	F	F	T	T	T	F
F	T	F	T	T	T	F	F	T	F	T
F	F	F	T	F	T	T	F	F	T	F
					$\triangle$					

Invalid, satisfiable

Part (iii)

p	q	$(p$	$\wedge$	$(q$	$\Leftrightarrow$	$\neg$	$q)$
T	T	T	F	T	F	F	T
T	F	T	F	F	F	T	F
F	T	F	F	T	F	F	T
F	F	F	F	F	F	T	F
			$\triangle$				

Invalid, unsatisfiable

Part (iv)

p	q	p	$\vee$	$\neg$	$(p$	$\wedge$	$q)$
T	T	T	T	F	T	T	T
T	F	T	T	T	T	F	F
F	T	F	T	T	F	F	T
F	F	F	T	T	F	F	F
			$\triangle$				

Valid, satisfiable

Part (v)

p	p	$\vee$	$\neg$	$(p$	$\wedge$	$p)$
T	T	T	F	T	T	T
F	F	T	T	F	F	F
		$\triangle$				

Valis, satisfiable

Part (vi)

p	q	$\neg$	$(\neg$	$(p$	$\wedge$	$q)$	$\wedge$	$\neg$	$(p$	$\vee$	$q))$
T	T	T	F	T	T	T	F	F	T	T	T
T	F	T	T	T	F	F	F	F	T	T	F
F	T	T	T	F	F	T	F	F	F	T	T
F	F	F	T	F	F	F	T	T	F	F	F
		$\triangle$									

Invalid, satisfiable

Part (vii)

p	q	r	$((p$	$\wedge$	$\neg$	$q)$	$\vee$	$(\neg$	p	$\wedge$	$q))$	$\Rightarrow$	r
T	T	T	T	F	F	T	F	F	T	F	T	T	T
T	T	F	T	F	F	T	F	F	T	F	T	T	F
T	F	T	T	T	T	F	T	F	T	F	F	T	T
T	F	F	T	T	T	F	T	F	T	F	F	F	F
F	T	T	F	F	F	T	T	T	F	T	T	T	T
F	T	F	F	F	F	T	T	T	F	T	T	F	F
F	F	T	F	F	T	F	F	T	F	F	F	T	T
F	F	F	F	F	T	F	F	T	F	F	F	T	F
												$\triangle$	

Invalid, satisfiable

Question 4

Part (i)

If valid, then the wff is T under every interpretation. Its negation will be F in every interpretation, hence unsatisfiable.

Part (ii)

If unsatisfiable, then the wff is F in every interpretation. Its negation will be T in every interpretation, hence valid.

Part (iii)

Option (a)

We're told $p \vee \neg p$ is valid, so its negation $\neg(p \vee \neg p)$ will be unsatisfiable and invalid.

Option (b)

We're told that $p \wedge \neg p$ is unsatisfiable, so its negation $\neg(p \wedge \neg p)$ will be valid and satisfiable.

Part (iv)

q	$(q$	$\Leftrightarrow$	$\neg$	$q)$	$\vee$	$\neg$	$(q$	$\Leftrightarrow$	$\neg$	$q)$
T	T	F	F	T	T	T	T	F	F	T
F	F	F	T	F	T	T	F	F	T	F
					$\triangle$					

Part (v)

Part (vi)

We can use the valid wff $p \Rightarrow p$ as the basis for a template:

For any wff $W, W \Rightarrow W$ is valid

Since $(p \wedge q) \Rightarrow (p \wedge q)$ matches this template (with $(p \wedge q)$ replacing W), it is valid (and hence is satisfiable).

Part (vii)

We can't use any of the templates we've been given so far, so (unless you find a template from elsewhere) we must draw up a truth table.

p	q	$\neg$	p	$\Rightarrow$	$(\neg$	p	$\vee$	$q)$
T	T	F	T	T	F	T	T	T
T	F	F	T	T	F	T	F	F
F	T	T	F	T	T	F	T	T
F	F	T	F	T	T	F	T	F
				$\triangle$				

Valid and satisfiable

Part (viii)

We can use the valid wff $\neg(p \wedge \neg p)$ as the basis for a template:

$$\text{For any wff } W, \neg(W \wedge \neg W) \text{ is valid}$$

Its negation is another template:

$$\text{For any wff } W, \neg\neg(W \wedge \neg W) \text{ is unsatisfiable}$$

Since $\neg\neg((p_1 \vee p_2) \wedge \neg(p_1 \vee p_2))$ matches this template (with $(p_1 \vee p_2)$ matching W), this wff is unsatisfiable (and hence invalid).

Part (ix)

Here's a crafty one. In part (vii) of this question, we established that $\neg p \Rightarrow (\neg p \vee q)$ was valid, and its negation can be used here as follows:

$$\text{For any wffs } W_1 \text{ and } W_2, \neg(\neg W_1 \Rightarrow (\neg W_1 \vee W_2)) \text{ is unsatisfiable}$$

Since $\neg(\neg(p_1 \Leftrightarrow p_2) \Rightarrow (\neg(p_1 \Leftrightarrow p_2) \vee (p_3 \wedge (p_2 \vee p_1))))$ macthes this template (with $p_1 \Leftrightarrow p_2$ replacing W_1 and $p_3 \wedge (p_2 \vee p_1)$ replacinging W_2), it is unsatisfiable (and hence invalid).

Question 5

Part (i)

p	q	$p \Leftrightarrow q$	q	p	
T	T	T	T	T	*
T	F	F	F	T	
F	T	F	T	F	
F	F	T	F	F	

In the only interpretation under which all the premisses are true (labelled with an asterisk), the conclusion is also true.

Part (ii)

p	q	r	$p \Rightarrow \neg q$	$\neg r \Rightarrow p$	q	r	
T	T	T	F	T	T	T	
T	T	F	F	T	T	F	
T	F	T	T	T	F	T	
T	F	F	T	T	F	F	
F	T	T	T	T	T	T	*
F	T	F	T	F	T	F	
F	F	T	T	T	F	T	
F	F	F	T	F	F	F	

In the only interpretation in which all the premises are true, the conclusion is also true.

Part (iii)

p	q	$p \Leftrightarrow \neg q$	$p \Rightarrow q$	
T	T	F	T	
T	F	T	F	*1
F	T	T	T	*2
F	F	F	T	

There is only one premise. There are two interpretations under which all the premisses (i.e. all one of them!) are true. In the one labelled *2, the conclusion is also true, but in the one labelled *2, the conclusion is false. So $p \Rightarrow q$ is not a logical consequence of $\{p \Leftrightarrow \neg q\}$.

Part (iv)

p	q	r	$(q \vee p) \Rightarrow r$	$q \Rightarrow r$	
T	T	T	T	T	*1
T	T	F	F	F	
T	F	T	T	T	*2
T	F	F	F	T	
F	T	T	T	T	*3
F	T	F	F	F	
F	F	T	T	T	*4
F	F	F	T	T	*5

In all five interpretations under which all the premisses (all one of them!) are true, the conclusion is also true.

Question 6

Part (i)

Yes, this holds. Indeed, for any wff W, $\{W\} \models W$, i.e. W is a logical consequence of itself since it is bound to be true as a conclusion whenever it is true as a premiss.

Part (ii)

No, this doesn't hold. Indeed, for any wff W, $\{\neg W\} \not\models W$, i.e. a wff is not a logical consequence of its negation since the negation as a conclusion will always be false whenever the premiss is true. (In fact, this doesn't quite hold for *any* W: if W is valid, then it will follow from its negation —see the next two parts of the question.)

Part (iii)

Yes, this holds.

A *valid* wff W is a logical consequence of absolutely any set of wffs A.

We know, for example, that $p \vee \neg p$ is valid so, e.g.:

$$\{(p \vee q) \Rightarrow (q \wedge \neg r), p \vee r\} \models p \vee \neg p \text{ as in the question, but also}\ldots$$

$$\{q\} \models p \vee \neg p$$

$$\{q, \neg q\} \models p \vee \neg p$$

$$\emptyset \models p \vee \neg p$$

and so on! (Draw truth tables to confirm these examples if you want.)

Why is this so? The definition of logical consequence requires that the conclusion be true whenever all the premisses are true. Since a valid wff is always true, it is obviously going to be true whenever all the premisses are true.

Part (iv)

Yes, this holds.

If a set of wffs A is such that *no* interpretation will make all the wffs in A true, then absolutely any wff W will be a logical consequence of that set of wffs A.

We know, for example, that there is no interpretation that will make all the wffs in the $\{p, \neg p\}$ true so, e.g.:

$\{p, \neg p\} \models (p \vee q) \Rightarrow (q \wedge \neg r)$ as in the question, but also…

$$\{p, \neg p\} \models q$$

$$\{p, \neg p\} \models \neg q$$

$$\{p, \neg p\} \models q \Rightarrow r$$

$$\{p, \neg p\} \models q \vee \neg q$$

$$\{p, \neg p\} \models p \wedge \neg p$$

and so on! (Draw truth tables to confirm these if you want.)

Why is this so? There is *no* interpretation that makes all the premisses true. The definition of logical consequence requires only that the conclusion be true *whenever* the premisses are all true. Since they never are all true, the definition holds trivially.

This can be summed up in the slogan:

> Anything follows from a contradiction.

Question 7

Part (i)

Option (a)

W_1	W_2	$\neg$	$(W_1$	$\Rightarrow$	$W_2)$	W_1	$\wedge$	$\neg$	W_2
T	T	F	T	T	T	T	F	F	T
T	F	T	T	F	F	T	T	T	F
F	T	F	F	T	T	F	F	F	T
F	F	F	F	T	F	F	F	T	F
		$\triangle$					$\triangle$		

Option (b)

W_1	W_2	W_1	$\Rightarrow$	W_2	$\neg$	W_2	$\Rightarrow$	$\neg$	W_1
T	T	T	T	T	F	T	T	F	T
T	F	T	F	F	T	F	F	F	T
F	T	F	T	T	F	T	T	T	F
F	F	F	T	F	T	F	T	T	F
			$\triangle$				$\triangle$		

Option (c)

W_1	W_2	W_3	$(W_1$	$\wedge$	$W_2)$	$\Rightarrow$	W_3	W_1	$\Rightarrow$	$(W_2$	$\Rightarrow$	$W_3)$
T	T	T	T	T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	F	T	F	T	F	F
T	F	T	T	F	F	T	T	T	T	F	T	T
T	F	F	T	F	F	T	F	T	T	F	T	F
F	T	T	F	F	T	T	T	F	T	T	T	T
F	T	F	F	F	T	T	F	F	T	T	F	F
F	F	T	F	F	F	T	T	F	T	F	T	T
F	F	F	F	F	F	T	F	F	T	F	T	F
						$\triangle$			$\triangle$			

Part (ii)

(a)

$$\begin{aligned}
\text{RHS} &\equiv W_1 \wedge \neg W_2 \\
&\equiv \neg\neg(W_1 \wedge \neg W_2) && \text{(involution)} \\
&\equiv \neg(\neg W_1 \vee \neg\neg W_2) && \text{(De Morgan)} \\
&\equiv \neg(\neg W_1 \vee W_2) && \text{(involution)} \\
&\equiv \neg(W_1 \Rightarrow W_2) && \text{(defn of conditional)} \\
&\equiv \text{LHS}
\end{aligned}$$

(b)

$$\begin{aligned}
\text{RHS} &\equiv \neg W_2 \Rightarrow \neg W_1 \\
&\equiv \neg\neg W_2 \vee \neg W_1 && \text{(defn of conditional)} \\
&\equiv W_2 \vee \neg W_1 && \text{(involution)} \\
&\equiv \neg W_1 \vee W_2 && \text{(commutativity of } \vee) \\
&\equiv W_1 \Rightarrow W_2 && \text{(defn of conditional)} \\
&\equiv \text{LHS}
\end{aligned}$$

(c)

$$
\begin{aligned}
\text{LHS} &\equiv (W_1 \wedge W_2) \Rightarrow W_3 \\
&\equiv \neg(W_1 \wedge W_2) \vee W_3 && \text{(defn of conditional)} \\
&\equiv (\neg W_1 \vee \neg W_2) \vee W_3 && \text{(De Morgan)} \\
&\equiv \neg W_1 \vee (\neg W_2 \vee W_3) && \text{(associativity of } \vee\text{)} \\
&\equiv \neg W_1 \vee (W_2 \Rightarrow W_3) && \text{(defn of conditional)} \\
&\equiv W_1 \Rightarrow (W_2 \Rightarrow W_3) && \text{(defn of conditional)} \\
&\equiv \text{RHS}
\end{aligned}
$$

Question 8

Part (i)

$$W_1 \vee W_2 \equiv \neg(\neg W_1 \wedge \neg W_2)$$

Check this with a truth table.

Part (ii)

$$W_1 \wedge W_2 \equiv \neg(\neg W_1 \vee \neg W_2)$$

Check this with a truth table.

Part (iii)

$$W_1 \vee W_2 \equiv \neg W_1 \Rightarrow W_2$$

$$W_1 \wedge W_2 \equiv \neg(W_1 \Rightarrow \neg W_2)$$

Check these with truth tables.

Part (iv)

We know that $\neg$ and $\vee$ on their own are sufficient, so all we need do is express these two in terms of $\Downarrow$:

$$\neg W \equiv W \Downarrow W$$

$$W_1 \vee W_2 \equiv (W_1 \Downarrow W_2) \Downarrow (W_1 \Downarrow W_2)$$

Check these with truth tables.

Question 9

Part (i)

Option (a)

1.	$p_1 \wedge p_2$	premiss
2.	$p_3 \wedge p_4$	premiss
3.	p_1	1,$\wedge$-E-L
4.	p_4	2,$\wedge$-E-R
5.	$p_1 \wedge p_4$	3,4,$\wedge$-I

Option (b)

1.	p	premiss
2.	$p \Rightarrow q$	premiss
3.	$q \Leftrightarrow r$	premiss
4.	q	1,2,$\Rightarrow$-E
5.	r	3,4,$\Leftrightarrow$-E-R
6.	$q \wedge r$	4,5,$\wedge$-I

Option (c)

1.	$\neg\neg q$	premiss
2.	q	1,$\neg$-E
3.	$q \vee r$	2,$\vee$-I

Option (d)

1.	$p_1 \wedge p_2$	premiss
2.	$p_1 \Rightarrow p_3$	premiss
3.	$p_2 \Rightarrow p_4$	premiss
4.	p_1	1,$\wedge$-E-l
5.	p_3	2,4,$\Rightarrow$-E
6.	p_2	1,$\wedge$-E-R
7.	p_4	3,6,$\Rightarrow$-E
8.	$p_3 \wedge p_4$	5,7,$\wedge$-I

Part (ii)

Option (a)

1.	$(p \wedge q) \Rightarrow r$	premiss
2.	p	premiss
3.	Subderivation	
3.1	q	assumption
3.2	$p \wedge q$	2,3.1,$\wedge$-I
3.3	r	1,3.2,$\Rightarrow$-E
4.	$q \Rightarrow r$	3,$\Rightarrow$-I

Option (b)

1.	p	premiss
2.	Subderivation	
2.1	q	assumption
2.2	p	1, repetition
3.	$q \Rightarrow p$	2,$\Rightarrow$-I

Part (iii)

Option (a)

1.	Subderivation	
1.1	r	assumption
1.2	Subderivation	
1.2.1	p	assumption
1.2.2	Subderivation	
1.2.2.1	q	assumption
1.2.2.2	r	1.1, repetition
1.2.3	$q \Rightarrow r$	1.2.2, $\Rightarrow$-I
1.3	$p \Rightarrow (q \Rightarrow r)$	1.2, $\Rightarrow$-I
2.	$r \Rightarrow (p \Rightarrow (q \Rightarrow r))$	1,$\Rightarrow$-I

Option (b)

1.	Subderivation	
1.1	$q \Rightarrow r$	assumption
1.2	Subderivation	
1.2.1	$p \Rightarrow q$	assumption
1.2.2	Subderivation	
1.2.2.1	p	assumption
1.2.2.2	q	1.2.1,1.2.2.1,$\Rightarrow$-E
1.2.2.3	r	1.1,1.2.2.2,$\Rightarrow$-E
1.2.3	$p \Rightarrow r$	1.2.2 $\Rightarrow$-I
1.3	$(p \Rightarrow q) \Rightarrow (p \Rightarrow r)$	1.2,$\Rightarrow$-I
2.	$(q \Rightarrow r) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$	1,$\Rightarrow$-I

Part (iv)

It is sound if, for any W_1, W_2 and W_3:

$$\{(W_1 \wedge W_2) \Rightarrow W_3\} \models W_1 \Rightarrow W_3$$

We can find out whether this is so by drawing up a truth table:

W_1	W_2	W_3	$(W_1 \wedge W_2) \Rightarrow W_3$	$W_1 \Rightarrow W_3$	
T	T	T	T	T	*1
T	T	F	F	F	
T	F	T	T	T	*2
T	F	F	T	F	*3
F	T	T	T	T	*4
F	T	F	T	T	*5
F	F	T	T	T	*6
F	F	F	T	T	*7

Seven interpretations make all the premises (all one of them) true. But under one of these seven interpretations (*3) the conclusion is false. So the logical consequence doesn't hold. So the inference rule is unsound.