MATHS FOR INFORMATION PROCESSING

Problem Sheet 3

1

(i) Give three difference between tuples and sets.

(ii) Enumerate the elements of the following set:

$$\{1, 2\} \times \{a, b, c\} \times \{\alpha, \beta\}$$

(iii) $S \triangleq \{1, 2\}$. Enumerate the elements of the following sets:

(a) $(S \times S) \times S$

(b) S^3

(c) $S \times \emptyset$

(d) $\emptyset^2$

(iv) *In general*, $A_1 \times A_2 \neq A_2 \times A_1$. While this is true *in general*, there *are* cases where $A_1 \times A_2 = A_2 \times A_1$. Can you give any?

2

$S_1 \triangleq \{1, 2, 3, 4, 5\}$ and $S_2 \triangleq \{4, 5, 6\}$.

$$REL : S_1 \leftrightarrow S_2$$

Here is the graph/extension of *REL* in matrix form:

	4	5	6
1	1	1	1
2	1	0	1
3	1	0	1
4	1	0	1
5	1	1	1

(i) State whether each of the following is true or false.

(a) 1 *REL* 4

(b) 2 *REL* 5

(c) 3 *REL* 4

(d) 4 *REL* 5

(e) 5 *REL* 4

(ii) Enumerate the following sets:

(a) $\{a_2 \mid 3\ \mathit{REL}\ a_2\}$

(b) $\{a_1 \mid a_1 \in S_1$ and there is at least one $a_2 \in S_2$ such that $\langle a_1, a_2 \rangle \notin \mathit{REL}\}$

3

Given $S_1 \triangleq \{1, 2, 3, 4\}$ and $S_2 \triangleq \{a, b, c\}$ and

$$\mathit{REL} : S_1 \leftrightarrow S_2$$

$$\mathit{REL} \triangleq \{\langle 1, b\rangle, \langle 1, c\rangle, \langle 3, b\rangle, \langle 4, a\rangle, \langle 4, c\rangle\}$$

(i) Draw *REL*'s arrow diagram.

(ii) Give *REL*'s matrix.

(iii) Draw *REL*'s coördinate diagram.

(iv) Give *Def(REL)* and *Ran(REL)*.

(v) Is *REL* total or partial, surjective or not, functional or not, and injective or not? Hence, is *REL* one-to-one, many-to-one, one-to-many or many-to-many? Is it a bijection?

4

Below we enumerate the graphs/extensions of some relations of type $\{1,2,3\} \leftrightarrow \{4,5,6\}$. State whether each relation is total or partial, surjective or not, functional or not, injective or not, one-to-one, many-to-one, one-to-many or many-to-many. Which, if any, are bijections?

(i) $\{\langle 1,4\rangle, \langle 2,4\rangle, \langle 3,4\rangle\}$

(ii) $\{\langle 1,4\rangle, \langle 1,5\rangle, \langle 1,6\rangle\}$

(iii) $\{\langle 1,4\rangle, \langle 2,4\rangle, \langle 2,5\rangle, \langle 2,6\rangle, \langle 3,6\rangle\}$

(iv) $\{\langle 1,6\rangle, \langle 2,5\rangle, \langle 3,4\rangle\}$

5

Let F, I, G, W, C, O, T and S be the countries France, Italy and Greece, the comestibles Wine, Cheese and Olives, and the supermarkets Tesburys and Sainscos, respectively. We define two relations as follows.

$$
\begin{array}{rcl}
\mathit{Countries} & \triangleq & \{F, I, G\} \\
\mathit{Produce} & \triangleq & \{W, C, O\} \\
\mathit{SMarkets} & \triangleq & \{T, S\} \\
\\
\mathit{Exports} & : & \mathit{Countries} \leftrightarrow \mathit{Produce} \\
\mathit{Exports} & \triangleq & \{\langle F, W\rangle, \langle F, C\rangle, \\
& & \qquad \langle I, W\rangle, \langle G, O\rangle\} \\
\\
\mathit{Stocks} & : & \mathit{SMarkets} \leftrightarrow \mathit{Produce} \\
\mathit{Stocks} & \triangleq & \{\langle T, W\rangle, \langle T, C\rangle, \\
& & \qquad \langle S, W\rangle, \langle S, C\rangle, \langle S, O\rangle\}
\end{array}
$$

You can think of these two relations as defining a (rather trivial) database.

(i) We now want to interrogate the database to find out (a) where items are stocked, (b) potential suppliers to the supermarkets, and (c) which supermarkets Greece supplies to. The expressions that achieve these three interrogations are given below. Your task is to 'evaluate' them.

(a) *Stocks*$^{-1}$

(b) *Stocks*$^{-1} \circ$ *Exports*

(c) *Im(Stocks*$^{-1} \circ$ *Exports, {G})*

(ii) Write expressions of your own which, if they were evaluated, would retrieve from the 'database' the following:

(a) which supermarkets stock cheese

(b) which countries export cheese.

6

In this question, you define your own operations on relations.

(i) Define an operation *Total* that takes in a relation R and returns true or false accoridng to whether R is a total relation or not. In a similar vein, define *Partial, Surjective* and *NotSurjective*. Harder to do is to define *Functional, NotFunctional, Injective, NotInjective, OneToOne, ManyToOne, OneToMany, ManyToMany* and *Bijective*. Try these ones if you can.

(Your answers will take the following form, e.g.:

$$Total(R) \triangleq \ldots$$

You may use English as well as symbolic expressions on the right hand side, but try to be as precise as possible.)

(ii) What three conditions must be satisfied for one binary relation to equal another?

(iii) Here is a definition of a relational union operation, *RU*. In the spirit of a similar operation from your DIS course, it requires that the two relations have equal domains and codomains, and it then simply unions their graphs/extensions.

For any binary relations R_1 and R_2 such that $Dom(R_1) = Dom(R_2)$ and $Cod(R_1) = Cod(R_2)$, then their relational union is the relation $RU(R_1, R_2)$ such that

$$\begin{aligned} Dom(RU(R_1, R_2)) &\triangleq Dom(R_1) \\ Cod(RU(R_1, R_2)) &\triangleq Cod(R_1) \\ RU(R_1, R_2) &\triangleq R_1 \cup R_2 \end{aligned}$$

(a) In a similar vein to the definition of *RU*, give a definition of relational difference *RD*.

(b) Consider the following assertion:

> *RU* and *RD* give us a way of 'updating' relations. We can insert new tuples using *RU* and remove tuples using *RD*.

In what ways do you think these operations are *not* like insertion and deletion commands?

(c) [*Hard; Optional*]

To the extent that *RU* and *RD* do give us a form of insertion and deletion, we ought to complete this question by defining a form of modification operation.

Given binary relations R_1 and R_2 with equal domains and codomains, we want a new relation $RM(R_1, R_2)$ in which tuples of the graph/extension of R_2 supplant *corresponding* tuples in the graph/extension of R_1.

Try to give a definition of *RM*.

(d) [*Hard; Optional*]

Suppose F_1 and F_2 are functional binary relations that are defined on the same domain and codomain. Under what conditions will their union, $RU(F_1, F_2)$, be functional? (State your answer as formally as posible.)

For a set A, one particular relation on A is the **identity relation**. (In fact, this relation is functional so the phrase **identity function** is more often used.) For a set A, this relation is denoted Id_A and is defined such that

$$Id_A : A \leftrightarrow A$$

$$Id_A \triangleq \{\langle a, a\rangle \mid a \in A\}$$

i.e. it relates each element to itself.

Example: for the set $S \triangleq \{0, 1, 2\}$, Id_S is the relation on S such that

$$Id_S = \{\langle 0, 0\rangle, \langle 1, 1\rangle, \langle 2, 2\rangle\}$$

This relation may seem irrelevant, but, just like the numbers 0 and 1 for addition and multiplication and the sets $\emptyset$ and U for $\cup$ and $\cap$, the identity relation on a set plays a role in Identity laws for relation composition.

For any relation $R : A_1 \leftrightarrow A_2$, which of the following four are true?

1. $R \circ Id_{A_1} = R$
2. $R \circ Id_{A_2} = R$
3. $Id_{A_1} \circ R = R$
4. $Id_{A_2} \circ R = R$

8

In this question, let *Parent* be the 'is the parent of' relation and *Sibling* be the 'is a sibling of' relation, both defined on the set of human beings (alive or dead).

Give natural English paraphrases for the following relations. (Example *Parent* $\circ$ *Sibling* is the 'is an aunt opr uncle of' relation.)

(i) $Parent^{-1}$

(ii) $Sibling^{-1}$

(iii) *Sibling* $\circ$ *Parent*

(iv) *Parent* $\circ$ *Parent*$^{-1}$

(v) *Parent*$^{-1}\circ$ *Parent*

(vi) *Sibling* $\circ$ *Sibling*

(vii) *Parent*$^{-1}\circ$ *Sibling*

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Let $N \triangleq \{1, 2, 3, 4, 6\}$ and let *Divs* be the relation on N as follows:

$$Divs : N \leftrightarrow N$$

$$Divs \triangleq \{\langle x, y\rangle \mid x \text{ divides } y\}$$

(To explain: x divides y when x 'goes into' y a whole number of times with no remainder.)

(i) Enumerate the tuples in the graph/extension of *Divs*.

(ii) Draw *Div*'s directed graph.

(iii) Give an intensional definition of the graph/extension of *Divs*$^{-1}$.

(iv) State whether *Divs* is (a) reflexive, non-reflexive or antireflexive, (b) symmetric, non-symmetric or antisymmetric, (c) transitive or not, (d) an equivalence relation or not, and (e) a strict partial order, a non-strict partial order, a strict total order, a non-strict total order or not an order. Do the same for *Divs*$^{-1}$.

10

(i) For any relation R, what is the relationship between the totality and surjectivity of R and the totality and surjectivity of R^{-1}.

(ii) For any equivalence relation R on a set A, argue that R^{-1} will also be an equivalence relation.

(iii) Let R_1 and R_2 be relations whose graphs/extensions are non-empty on a set A that contains at least three elements.

State whether each of the following statements is true or false. If false, give a counter-example using the set $A \triangleq \{1, 2, 3\}$.

(a) If R_1 and R_2 are symmetric then the relation on A whose graph/extension is $R_1 \cap R_2$ is symmetric.

(b) If R_1 and R_2 are symmetric then the relation on A whose graph/extension is $R_1 \cup R_2$ is symmetric.

(c) If R_1 and R_2 are reflexive then the relation on A whose graph/extension is $R_1 \cap R_2$ is reflexive.

(d) If R_1 and R_2 are reflexive then the relation on A whose graph/extension is $R_1 \cup R_2$ is reflexive.

(e) If R_1 and R_2 are transitive then the relation on A whose graph/extension is $R_1 \cup R_2$ is transitive.

(f) If R_1 is reflexive then the relation on A whose graph/extension is $R_1 \cap R_1^{-1}$ is non-empty.

(g) If R_1 is symmetric then the relation on A whose graph/extension is $R_1 \cap R_1^{-1}$ is non-empty.

11

(i) Let C be a set of car models,

$$C \triangleq \{\textit{Fiesta, Escort, Quattro, Polo, Uno}\}$$

For the sake of the following question, assume the following information:

Model:	*Fiesta*	*Escort*	*Quattro*	*Polo*	*Uno*
Manufacturer:	*Ford*	*Ford*	*Audi*	*VW*	*Fiat*
Cost:	7000	8000	15000	8000	7500

For both of the following relations, say whether the relation is an equivalence relation. If you say it is, say why, and give the equivalence classes (i.e. the partition of C). If you say it is not, say why not.

(a) *MET* is the relation on C such that c_1 *MET* c_2 if car model c_1 is more expensive than car model c_2.

(b) *SMA* is the relation on C such that c_1 *SMA* c_2 if car model c_1 is made by the same manufacturer as car model c_2.

(ii) Determine whether each of the following relations on sets is (a) reflexive, non-reflexive or antireflexive, (b) symmetric, non-symmetric or antisymmetric, (c) transitive or not, (e) an equivalence relation or not, and (f) a strict partial order, a non-strict partial order, a strict total order, a non-strict total order or not an order.

(a) $\leq$ on $\mathbb{N}$

(b) *RhymesWith* on the set of English words

(c) *NextDoorTo* on the set of buildings

(d) *AncestorOf* on the set of people

(e) *Before* on the set of times.

(iii) Take any non-empty set U. Consider the binary relation *Uni* defined on U as follows:

$$Uni : U \leftrightarrow U$$

$$Uni \triangleq U^2$$

Is *Uni* (a) reflexive, non-reflexive or antireflexive, (b) symmetric, non-symmetric or antisymmetric, (c) transitive or not, (d) an equivalence relation or not, (e) an order or not. If you say it is an equivalence relation give the partition. If you say it is an order, say what type of order it is.

(iv) [*Harder; Optional but you're encouraged to try this one*]

Repeat the previous question for the following relation:

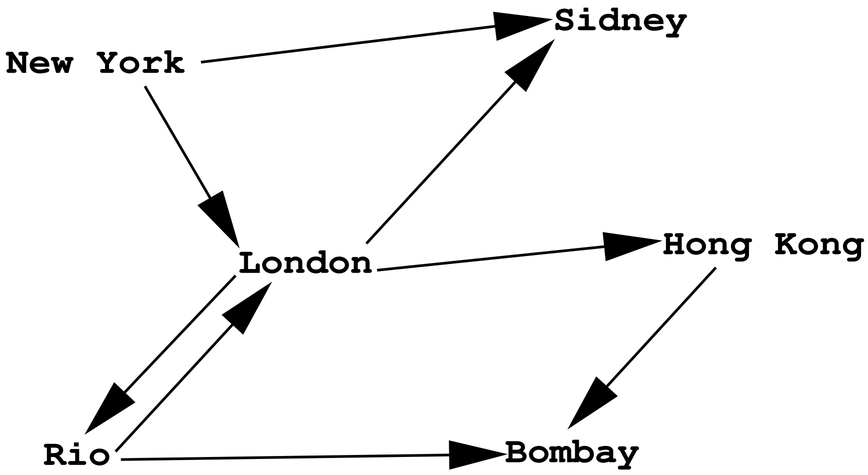
$$Emp : U \leftrightarrow U$$

$$Emp \triangleq \emptyset$$

12

This is another question in which we specify a trivial 'database' and write expressions for interrogating the database.

Consider the following network of flights flown by MIPAir.



We'll use capital letters to represent the cities. So the set of cities is defined as follows:

$$Cities \triangleq \{N, S, L, R, H, B\}$$

(i) Specify the *Flights* relation. (The flights have no stop-overs.)

(ii) How many (non-stop-over) flights paths are flown?

(iii) Expansion is anticipated: we would like every city to have a direct flight to every other city. How many flights paths would we then fly?

(iv) Paraphrase the following expressions into English that is as natural as possible and then 'evaluate' them.

(a) *Ran(Flights)*

(b) *Def(Flights)*

(c) *TC(Flights)*

(v) Write expressions that apply operations to *Cities* and/or *Flights* that, if they were evaluated, would tell us

(a) which cities one cannot fly to

(b) which cities one cannot fly out of

(c) which cities can be reached directly from London

(d) which cities can be reached from either New York or London with one stopover

(e) from which cities one can fly *directly* to Bombay

(f) from which cities one can fly to Bombay (both directly and non-directly).

(vi) Now go back to you answer from part (v) and evaluate all the expressions you have written.

13

State whether the following functions are total or partial, surjective or not, and injective or not.

(i) $f_1 : \mathbb{N} \to \mathbb{N}$ where $f_1(x) \triangleq x + 1$

(ii) $f_2 : \mathbb{Z} \to \mathbb{Z}$ where $f_2(x) \triangleq x + 1$

(iii) $f_3 : \mathbb{Z} \to \mathbb{Z}$ where $f_3(x) \triangleq \mathit{abs}(x) + 1$ given that $\mathit{abs}(x)$ turns a negative number positive and leaves a postive number unchanged, e.g. $\mathit{abs}(3) = 3; \mathit{abs}(-3) = 3$.

14

(i) Is the compositions of two functions also a function? Explain your answer.

(ii) The inverse of a function is not necessarily a function. When is the inverse of a function also a function?

(iii) When is the inverse of a function a bijection?

15

(i) Of the dishes Cheese Omelette (*CO*), Devilled Kidneys (*DK*), Liver and Onions (*LO*), Swiss Foundue (*SF*) and Eggs Florentine (*EF*), I'm interested in only the following set:

$$\mathit{Dishes} \triangleq \{\mathit{CO}, \mathit{SF}, \mathit{EF}\}$$

Represent this set as a function.

(ii) Cheese Omelettes require eggs and cheese, Swiss Foundue requires cheese and wine, and Eggs Florentine requires eggs, cheese and spinach. Using the obvious abbreviations for these ingredients, we have:

$$\begin{array}{rcl} \mathit{Ingredients} & \triangleq & \{E, C, W, S\} \\ \\ \mathit{Recipe} & : & \mathit{Dishes} \leftrightarrow \mathit{Ingredients} \\ \mathit{Recipe} & \triangleq & \{\langle CO, E\rangle, \langle CO, C\rangle, \\ & & \langle SF, C\rangle, \langle SF, W\rangle, \\ & & \langle EF, E\rangle, \langle EF, C\rangle, \langle EF, S\rangle\} \end{array}$$

This relation is non-functional. Specify two different *functions* that represent the same information.

(iii) I decide to have Eggs Florentine, which requires its ingredients in the order eggs, then spinach, then cheese for the sauce. This ordering we can represent using a tuple:

$$\langle E, S, C\rangle$$

Represent this tuple as a function.

(iv) I look in my fridge and I find 3 eggs, 0 lumps of chesse and a dead mouse.

This is not a set as it contains duplicated items.

To model it as a tuple (e.g. $\langle E1, E2, E3, M\rangle$, where $E1, E2$ and $E3$ are the eggs and M is the mouse) imposes an ordering, which seems unnatural.

Mathematicans use a new kind of mathematical object, the **bag**, to model this kind of scenario. (Bags are just like sets, but they allow duplicated items.) But we don't follow them down that route. The contents of my fridge are easily represented as a function. (Indeed, all bags can be represented as functions.)

Can you specify this function?

(v) I give up on this cooking lark. What's the telephone number of the Chinese takeaway?

16 [*Hard; Optional, but you're encouraged to have a go at it*]

(i) When we introduce relations we write, e.g.

$$R : A_1 \leftrightarrow A_2$$

for a relation R with domain A_1 and codomain A_2. In this question, we explore what the symbols ':' and '$\leftrightarrow$' mean.

The symbol ':' means '$\in$'. The reason we use ':' and not '$\in$' is like the reason we use '$\triangleq$' and not '$=$' in certain cases: '$\triangleq$' and ':' are definitional: using ':' indicates that R is *defined* to be a member of some set.

Now, that means that $A_1 \leftrightarrow A_2$ must be a set.

So, $\leftrightarrow$ is an operation that takes in two sets, A_1 and A_2, and returns a third set (a set of which R is a member). Given that you know what the graph/extension of a particular relation is like, and that $A_1 \leftrightarrow A_2$ is a set of these, state as formally as possible exactly what '$\leftrightarrow$' does.

(ii) Use your answer to part (i) to give an expression that would compute how many different relations there are from a set A_1 to a set A_2 in general.

17 [*Hard; Optional, but you're encouraged to have a go at it*]

In this question, we give some type specifications for relations and functions. Your task is to paraphrase them into English and then give an example (from this section of the course, the previous sections of the course, or one you make up) of a relation/function of that type.

Example:

$$\mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

Paraphrase: a binary function from pairs of natural numbers to natural numbers. Example: addition on natural numbers.

(i) $\mathbb{N} \to \mathbb{N}$

(ii) $\wp(A) \to \mathbb{N}$

(iii) $\wp(A_1) \times \wp(A_2) \to \wp(A_1)$

(iv) $\wp(A_1) \times \wp(A_2) \to \wp(A_1 \times A_2)$

(v) $(A_1 \leftrightarrow A_2) \to Bool$

(vi) $(A_1 \leftrightarrow A_2) \to (A_2 \leftrightarrow A_1)$

(vii) $(A_1 \leftrightarrow A_2) \times (A_2 \leftrightarrow A_3) \to (A_1 \leftrightarrow A_3)$

(viii) $(A_1 \leftrightarrow A_2) \times \wp(A_1) \to \wp(A_2)$

(ix) $\wp(A) \to (A \to Bool)$

(x) $(A \to Bool) \to \wp(A)$

18 [Optional: for the mathematically inclined]

We saw that the graph/extension of a relation can be expressed as a matrix. Certain operations on matrices will correspond to useful operations on relations. Look through maths textbooks to find matrix operations, and list those you think do correspond to useful operations on relations.

Model Answers

MATHS FOR INFORMATION PROCESSING

Problem Sheet 3

Question 1

Part (i)

A tuple has a fixed no. of elements; a set does not. A tuple can have repeated elements; a set cannot. The elements in a tuple are ordered; those in a set are not.

Part (ii)

$\{\langle 1, a, \alpha\rangle, \langle 1, a, \beta\rangle, \langle 2, a, \alpha\rangle, \langle 2, a, \beta\rangle, \langle 1, b, \alpha\rangle, \langle 1, b, \beta\rangle,$
$\langle 2, b, \alpha\rangle, \langle 2, b, \beta\rangle, \langle 1, c, \alpha\rangle, \langle 1, c, \beta\rangle, \langle 2, c, \alpha\rangle, \langle 2, c, \beta\rangle\}$

Part (iii)
Option (a)

$\{\langle\langle 1,1\rangle,1\rangle, \langle\langle 1,1\rangle,2\rangle, \langle\langle 1,2\rangle,1\rangle, \langle\langle 1,2\rangle,2\rangle, \langle\langle 2,1\rangle,1\rangle, \langle\langle 2,1\rangle,2\rangle, \langle\langle 2,2\rangle,1\rangle, \langle\langle 2,2\rangle,2\rangle\}$

Option (b)

$\{\langle 1,1,1\rangle, \langle 1,1,2\rangle, \langle 1,2,1\rangle, \langle 1,2,2\rangle, \langle 2,1,1\rangle, \langle 2,1,2\rangle, \langle 2,2,1\rangle, \langle 2,2,2\rangle\}$

Option (c)

$\emptyset$

Option (d)

$\emptyset$

Part (iv)

When $A_1 = A_2$, then $A_1 \times A_2 = A_2 \times A_1$.

When $A_1 = \emptyset$ or $A_2 = \emptyset$ (or both), then $A_1 \times A_2 = A_2 \times A_2 = \emptyset$.

Question 2

Part (i)
Option (a)

True

Option (b)

False

Option (c)

True

Option (d)

False

Option (e)

True

Part (ii)

Option (a)

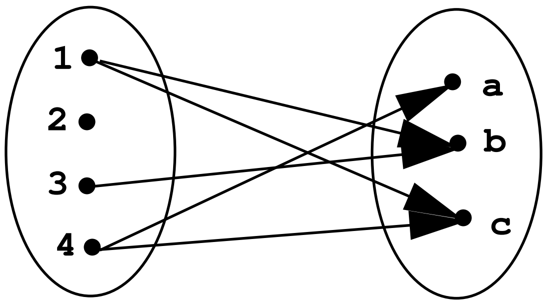
$\{4, 6\}$

Option (b)

$\{2, 3, 4\}$

Question 3

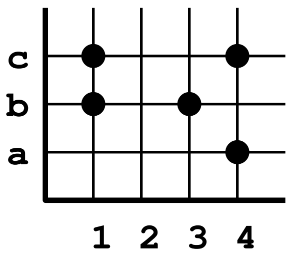
Part (i)



Part (ii)

	a	b	c
1	0	1	1
2	0	0	0
3	0	1	0
4	1	0	1

Part (iii)



Part (iv)

$$Def(REL) = \{1, 3, 4\}$$
$$Ran(REL) = \{a, b, c\}$$

Part (v)

Partial, surjective, not functional, not injective, many-to-many and not a bijection.

Question 4

Part (i)

Total, not surjective, functional, not injective, many-to-one

Part (ii)

Partial, surjective, not functional, injective, one-to-many

Part (iii)

Total, surjective, not functional, not injective, many-to-many

Part (iv)

Total, surjective, functional, injective, one-to-one. This is a bijection.

Question 5

Part (i)
Option (a)

$$
\begin{array}{rcl}
\textit{Stocks}^{-1} & : & \textit{Produce} \leftrightarrow \textit{SMarkets} \\
\textit{Stocks}^{-1} & = & \{\langle W,T\rangle,\langle C,T\rangle, \\
 & & \langle W,S\rangle,\langle C,S\rangle,\langle O,S\rangle
\end{array}
$$

Option (b)

$$
\begin{array}{rcl}
\textit{Stocks}^{-1} \circ \textit{Exports} & : & \textit{Countries} \leftrightarrow \textit{SMarkets} \\
\textit{Stocks}^{-1} \circ \textit{Exports} & = & \{\langle F,T\rangle,\langle I,T\rangle, \\
 & & \langle F,S\rangle,\langle I,S\rangle,\langle G,S\rangle\}
\end{array}
$$

Option (c)

$\textit{Im(Stocks}^{-1} \circ \textit{Exports, \{G\})} = \{S\}$

Part (ii)

Option (a)

$\textit{Im(Stocks}^{-1}\textit{, \{C\})}$

Option (b)

$\textit{Im(Exports}^{-1}\textit{, \{C\})}$

Question 6

Part (i)

$$
\begin{aligned}
\textit{Total}(R) &\triangleq \textit{Def}(R) = \textit{Dom}(R)\\
\textit{Partial}(R) &\triangleq \textit{Def}(R) \subset \textit{Dom}(R)\\
\textit{Surjective}(R) &\triangleq \textit{Ran}(R) = \textit{Cod}(R)\\
\textit{NotSurjective}(R) &\triangleq \textit{Ran}(R) \subset \textit{Cod}(R)\\
\textit{Functional}(R) &\triangleq \text{there does not exist an } a_1\\
&\quad\ \text{such that there exists an } a_2 \text{ and an } a_3\\
&\quad\ \text{where } a_2 \neq a_3 \text{ for which } \langle a_1, a_2\rangle \in R\\
&\quad\ \text{and } \langle a_1, a_3\rangle \in R.\\
\textit{NotFunctional}(R) &\triangleq \text{there does exist an } a_1\\
&\quad\ \text{such that there exists an } a_2 \text{ and an } a_3\\
&\quad\ \text{where } a_2 \neq a_3 \text{ for which } \langle a_1, a_2\rangle \in R\\
&\quad\ \text{and } \langle a_1, a_3\rangle \in R.\\
\textit{Injective}(R) &\triangleq \text{there does not exist an } a_1\\
&\quad\ \text{such that there exists an } a_2 \text{ and an } a_3\\
&\quad\ \text{where } a_2 \neq a_3 \text{ for which } \langle a_2, a_1\rangle \in R\\
&\quad\ \text{and } \langle a_3, a_1\rangle \in R.\\
\textit{NotInjective}(R) &\triangleq \text{there does exist an } a_1\\
&\quad\ \text{such that there exists an } a_2 \text{ and an } a_3\\
&\quad\ \text{where } a_2 \neq a_3 \text{ for which } \langle a_2, a_1\rangle \in R\\
&\quad\ \text{and } \langle a_3, a_1\rangle \in R.\\
\textit{OneToOne}(R) &\triangleq \textit{Functional}(R) \text{ and } \textit{Injective}(R)\\
\textit{ManyToOne}(R) &\triangleq \textit{Functional}(R) \text{ and } \textit{NotInjective}(R)\\
\textit{OneToMany}(R) &\triangleq \textit{NotFunctional}(R) \text{ and } \textit{Injective}(R)\\
\textit{ManyToMany}(R) &\triangleq \textit{NotFunctional}(R) \text{ and } \textit{NotInjective}(R)\\
\textit{Bijective}(R) &\triangleq \textit{OneToOne}(R) \text{ and } \textit{Total}(R) \text{ and } \textit{Surjective}(R)
\end{aligned}
$$

Part (ii)

The two relations must have the same (a) domain, (b) codomain, and (c) graph/extension.

Part (iii)
Option (a)

For any binary relations R_1 and R_2 such that *Dom*(R_1) = *Dom*(R_2) and *Cod*(R_1) = *Cod*(R_2), then their relational difference is the relation *RD*(R_1, R_2) such that

$$
\begin{aligned}
\textit{Dom}(\textit{RD}(R_1, R_2)) &\triangleq \textit{Dom}(R_1)\\
\textit{Cod}(\textit{RD}(R_1, R_2)) &\triangleq \textit{Cod}(R_1)\\
\textit{RD}(R_1, R_2) &\triangleq R_1 \setminus R_2
\end{aligned}
$$

Option (b)

Firstly, they do not change the relation. Instead they form a different relation as their result. Secondly, with an insertion command, if you were to insert something that was already in the relation, you would expect an error message; but with union, you are allowed to union in things that are already there without complaint. Similarly, a typical deletion command would complain if you tried to delete tuples that weren't in the relation, but the relational difference operation will not complain.

Option (c)

For any binary relations R_1 and R_2 such that $Dom(R_1) = Dom(R_2)$ and $Cod(R_1) = Cod(R_2)$, then the relational modification of R_1 by R_2 is the relation $RM(R_1, R_2)$ such that

$$\begin{aligned} Dom(RM(R_1, R_2)) &\triangleq Dom(R_1) \\ Cod(RM(R_1, R_2)) &\triangleq Cod(R_1) \\ RM(R_1, R_2) &\triangleq (((Def(R_1) \setminus Def(R_2)) \times Cod(R_1)) \cap R_1) \cup R_2 \end{aligned}$$

Option (d)

When $Def(F_1) \cap Def(F_2) = \emptyset$: i.e. when they have disjoint definition domains, that way unioning their graphs/extensions will not introduce maplets that have more than one destination for a particular domain value.

In fact, the above answer is more restrictive than necessary: they could have definition domain values in common, as long as those values map to the same codomain value. So a better answer (though much harder to arrive at) is: when $((Def(F_1) \cap Def(F_2)) \times Cod(F_1)) \cap F_1 = ((Def(F_1) \cap Def(F_2)) \times Cod(F_1)) \cap F_2$.

Question 7

The first and fourth are true.

Question 8

Part (i)

'is a child of'

Part (ii)

'is a sibling of'

Part (iii)

'is the parent of a sibling of', i.e. 'is the parent of' (since your sibling's parent is your parent)

Part (iv)

'is the same as' (identity function)

Part (v)

'is the same as' (identity function)

Part (vi)

'is a sibling of' (since your sibling's sibling is your sibling)

Part (vii)

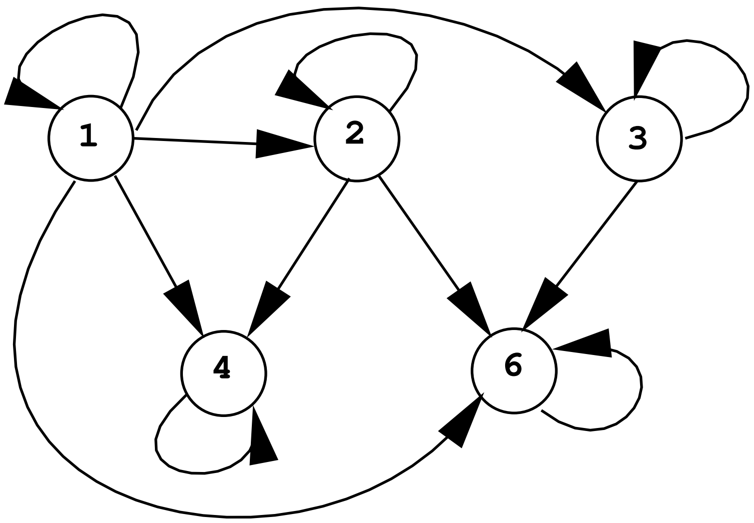
'is the sibling of the child of', i.e. 'is a child of' (since your sibling's parent is your parent)

Question 9

Part (i)

$\{\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 1,4\rangle, \langle 1,6\rangle, \langle 2,2\rangle, \langle 2,4\rangle, \langle 2,6\rangle, \langle 3,3\rangle, \langle 3,6\rangle, \langle 4,4\rangle, \langle 6,6\rangle\}$

Part (ii)



Part (iii)

Divs$^{-1} = \{\langle x, y\rangle \mid x$ is a multiple of $y\}$ or *Divs*$^{-1} = \{\langle x, y\rangle \mid y$ divides $x\}$

Part (iv)

Divs is (a) reflexive, (b) antisymmetric, (c) transitive, (d) not an equivalence relation, and (e) a non-strict partial order.

Divs^{-1} is the same. (This should have been obvious: if you reverse the direction of *all* of the arrows in *Div*'s directed graph, to give *Div*$^{-1}$, this won't change any of these properties; only adding or deleting arrows can change the properties. See question (10), part (ii) on this matter.)

Question 10

Part (i)

If R is total then R^{-1} is surjective; if R is partial then R^{-1} is not surjective; if R is surjective, then R^{-1} is total; if R is not surjective, then R^{-1} is partial.

Part (ii)

We know that for any pair of objects a_1 and a_2 such that $a_1 \; R \; a_2$ then $a_2 \; R^{-1} \; a_1$. And that if R is an equivalence relation, then it is reflexive, symmetric and transitive. We must argue that R^{-1} is also reflexive, symmetric and transitive.

If R is reflexive, then $a \; R \; a$ for all $a \in A$. If $a \; R \; a$, then $a \; R^{-1} \; a$ (swapping the objects around having no discernible effect here!). So R^{-1} is also reflexive.

If R is symmetric, then where $a_1 \; R \; a_2$ then also $a_2 \; R \; a_1$. In cases where $a_1 \; R \; a_2$ and $a_2 \; R \; a_1$, then $a_2 \; R^{-1} \; a_1$ and $a_1 \; R^{-1} \; a_2$, which makes R^{-1} symmetric too.

If R is transitive, then where $a_1 \; R \; a_2$ and $a_2 \; R \; a_3$ then also $a_1 \; R \; a_3$. In cases where, $a_1 \; R \; a_2$, $a_2 \; R \; a_3$ and $a_1 \; R \; a_3$, then $a_2 \; R^{-1} \; a_1$, $a_3 \; R^{-1} a_2$ and $a_3 \; R^{-1} \; a_1$, which makes R^{-1} transitive too.

Part (iii)

Option (a)

True

Option (b)

True

Option (c)

True

Option (d)

True

Option (e)

False: suppose $1 \; R_1 \; 2$ only and $2 \; R_2 \; 3$ only

Option (f)

True

Option (g)

True

Question 11

Part (i)

Option (a)

No, it is not an equivalence relation. To be an equivalence relation it must be reflexive, symmetric and transitive. It is neither reflexive nor symmetric. In no case is c is more expensive than c so it is not reflexive (in fact it's antireflexive). In fact, that's all one needs to say. To be an equivalence relation it must satisfy the three properties. We have shown it doesn't satisfy the first, so whether it satisfies the other two doesn't really matter. So we could stop at this point. As a matter of academic interest I will show that it is not symmetric as well. For $c_1 \neq c_2$, if c_1 is more expensive than c_2, it is not the case that c_2 is more expensive than c_1, so it is not symmetric (in fact it is antisymmetric). (It is transitive which I'll let you confirm for yourself. Note that as it is antisymmetric and transitive this is an ordering, and since it is reflexive it is non-strict.)

(The above might be more clearly seen by drawing the relation's directed graph.)

Option (b)

Yes, it is an equivalence relation. We must show that it is reflexive, symmetric and transitive. Reflexive: for any c, clearly c is made by the same manufacturer as c. Symmetric: for any $c_1, c_2, c_1 \neq c_2$, if c_1 is made by the same manufacturer as c_2, then, clearly, c_2 is made by the same manufacturer as c_1. Transitive: for any $c_1, c_2, c_3, c_1 \neq c_2 \neq c_3$, if c_1 is made by the same manufacturer as c_2 and c_2 is made by the same manufacture as c_3, then c_1 is clearly made by the same manufacturer as c_3. The equivalence classes shown as a partition are:

{{Fiesta, Escort}, {Quattro}, {Polo}, {Uno}}

(The above might be more clearly seen by drawing the relation's directed graph.)

Part (ii)

Option (a)

Reflexive, antisymmetric, transitive, not an equivalence relation, non-strict total order.

Option (b)

Reflexive, symmetric, transitive, equivalence relation, not an order.

Option (c)

Antireflexive, symmetric, not transitive, not an equivalence relation, not an order

Option (d)

Antireflexive, antisymmetric, transitive, not an equivalence relation, strict total order.

Option (e)

Antireflexive, antisymmetric, transitive, not an equivalence relation, strict total order.

Part (iii)

Since everything is related to everything, the relation is reflexive, symmetric, and transitive. So it is an equivalence relation. The partition is $\{U\}$, i.e. there is only one equivalence class within the partition, and this class contains all the items of U.

Uni is not an order, because it is not antisymmetric.

Part (iv)

This is antireflexive (since it is not the case that every item in U, for non-empty U, is related to itself).

It is symmetric (because, vacuously, if a pair is in the graph/extension so is its reverse — it's just that there are no pairs in the graph/extension). But it is also antisymmetric (because, vacuously, if a pair is in the graph/extension, it's reverse isn't in the graph/extension — again there are no pairs)!!

Similarly, it is transitive.

So, although it is symmetric and transitive, it is not reflexive, so it is not an equivalence relation. But it is antisymmetric and transitive, so it is an ordering. Being antireflxive, it is strict, and since there are no incomparable elements, it is total.

By the way, if U was empty, then *Emp* would be reflexive (as well as antireflexive), and so it would additionally be an equivalence relation, whose partition would be $\{\emptyset\}$.

Question 12

Part (i)

$$\mathit{Flights} : \mathit{Cities} \leftrightarrow \mathit{Cities}$$

$$\mathit{Flights} \triangleq \{\langle N, S\rangle, \langle N, L\rangle, \langle L, S\rangle, \langle L, R\rangle, \langle R, L\rangle, \langle R, B\rangle, \langle L, H\rangle, \langle H, B\rangle\}$$

Part (ii)

The cardinality of the graph/extension: 8

Part (iii)

card(Cities $\times$ *Cities)* $=$ *card(Cities)* $*$ *card(Cities)* $= 6 * 6 = 36$. But, from this we have to subtract 6, because Cities $\times$ Cities includes flights direct from a city to itself. (The question says there will be flight from every city to every *other* city. That little word *other* is crucial!) So the answer is $36 - 6 = 30$.

Part (iv)

Option (a)

Cities we have direct flights to.

$\{B, H, L, R, S\}$

Option (b)

Cities we have direct flights from.

$\{H, L, N, R\}$

Option (c)

The relation whose graph/extension shows connections between one city and another by flights that are both direct and non-direct.

$$\begin{aligned}
\mathit{TC(Flights)} &: \quad \mathit{Cities} \leftrightarrow \mathit{Cities} \\
\mathit{TC(Flights)} &= \quad \{\langle N, S\rangle, \langle N, L\rangle, \langle L, S\rangle, \\
&\qquad \langle L, R\rangle, \langle R, L\rangle, \langle R, B\rangle, \\
&\qquad \langle L, H\rangle, \langle H, B\rangle, \langle N, B\rangle, \\
&\qquad \langle N, H\rangle, \langle N, R\rangle, \langle L, B\rangle, \\
&\qquad \langle R, H\rangle, \langle R, S\rangle\}
\end{aligned}$$

Part (v)

Option (a)

$\mathit{Cities} \setminus \mathit{Ran(Flights)}$

Option (b)

$\mathit{Cities} \setminus \mathit{Def(Flights)}$

Option (c)

$\mathit{Im(Flights, \{L\})}$

Option (d)

$\mathit{Im(Flights \circ Flights, \{N, L\})}$

Option (e)

$\mathit{Im(Flights^{-1}, \{B\})}$

Option (f)

$\mathit{Im(TC(Flights^{-1}), \{B\})}$

Part (vi)

$\{N\}$

$\{B, S\}$

$\{H, R, S\}$

$\{B, H, L, R, S\}$

$\{H, R\}$

$\{H, L, N, R\}$

Question 13

Part (i)

Total, not surjective, injective

Part (ii)

Total, surjective, injective

Part (iii)

Total, not surjective, not injective

Question 14

Part (i)

Yes it is. To explain this, we will show that if we have two functions F_1 and F_2, then it is impossible for thier composition $F_2 \circ F_1$ to be anything other than functional. If $F_2 \circ F_1$ is not functional, then there'll be at least one case where $F_2 \circ F_1$ maps a_1 to c_i and c_j ($i \neq j$). But the only ways for this to happen would be if F_1 weren't a function (e.g. it mapped a_1 to b_i and b_j, $i \neq j$ and then F_2 could map b_i to c_i and b_j to c_j) or if F_2 weren't a function (e.g. F_1 maps a_1 to b_1 and F_2 maps b_1 to c_i and c_j, $i \neq j$) (or of neither were functions).

Part (ii)

When the function is also injective. In other words, when the function is one-to-one, the inverse will be a function, but when the function is many-to-one, it won't.

Part (iii)

When the original function is a bijection.

Question 15

Part (i)

We need the characteristic function of the set *Dishes*, taking the universal set to be *{CO, DK, LO, SF, EF}*:

$$
\begin{array}{rcl}
CF_{Dishes} & : & Dishes \rightarrow Bool \\
\\
CF_{Dishes}(CO) & = & T \\
CF_{Dishes}(DK) & = & F \\
CF_{Dishes}(LO) & = & F \\
CF_{Dishes}(SF) & = & T \\
CF_{Dishes}(EF) & = & T
\end{array}
$$

Part (ii)

The first function uses a mapping to the powerset of the ingredients:

$$
\begin{array}{rcl}
RF & : & Dishes \rightarrow \wp(Ingredients) \\
\\
RF(CO) & = & \{E, C\} \\
RF(SF) & = & \{C, W\} \\
RF(EF) & = & \{E, C, S\}
\end{array}
$$

The second function uses the characteristic function of the graph/extension of the original relation.

$$
\begin{array}{rcl}
CF_{Recipe} & : & Dishes \times Ingredients \rightarrow Bool \\
\\
CF_{Recipe}(CO, E) & = & T \\
CF_{Recipe}(CO, C) & = & T \\
CF_{Recipe}(CO, W) & = & F \\
CF_{Recipe}(CO, S) & = & F \\
CF_{Recipe}(SF, E) & = & F \\
CF_{Recipe}(SF, C) & = & T \\
CF_{Recipe}(SF, W) & = & T \\
CF_{Recipe}(SF, S) & = & F \\
CF_{Recipe}(EF, E) & = & T \\
CF_{Recipe}(EF, C) & = & T \\
CF_{Recipe}(EF, W) & = & F \\
CF_{Recipe}(EF, S) & = & T
\end{array}
$$

Part (iii)

$$
\begin{array}{rcl}
EFT & : & \mathbb{P} \rightarrow Ingredients \\
\\
EFT(1) & = & E \\
EFT(2) & = & S \\
EFT(3) & = & C
\end{array}
$$

(This is a partial function because not all positive integers are mapped, and it is not surjective because wine is not mapped to. We could make it surjective by choosing $\{E, S, C\}$ as the codomain.)

Part (iv)

$$
\begin{array}{rcl}
FC & : & \{E, C, M\} \rightarrow \mathbb{N} \\
\\
FC(E) & = & 3 \\
FC(C) & = & 0 \\
FC(M) & = & 1
\end{array}
$$

Part (v)

(01904) 638374

Question 16 [*Hard; Optional, but you're encouraged to have a go at it*]

Part (i)

$$A_1 \leftrightarrow A_2 \triangleq \wp(A_1 \times A_2)$$

i.e it takes two sets A_1 and A_2, forms their Cartesian product and then the power set of that product.

To illustrate this, consider $S_1 \triangleq \{1, 2\}$ and $S_2 \triangleq \{a, b\}$.

$$\begin{array}{rcl} S_1 \leftrightarrow S_2 & = & \wp(\{1,2\} \times \{a,b\}) \\ & = & \wp(\{\langle 1,a\rangle, \langle 1,b\rangle, \langle 2,a\rangle, \langle 2,b\rangle\}) \\ & = & \{\emptyset, \{\langle 1,a\rangle\}, \{\langle 1,b\rangle\}, \{\langle 2,a\rangle\}, \{\langle 2,b\rangle\}, \\ & & \{\langle 1,a\rangle, \langle 1,b\rangle\}, \{\langle 1,a\rangle, \langle 2,a\rangle\}, \\ & & \{\langle 1,a\rangle, \langle 2,b\rangle\}, \{\langle 1,b\rangle, \langle 2,a\rangle\}, \\ & & \{\langle 1,b\rangle, \langle 2,b\rangle\}, \{\langle 2,a\rangle, \langle 2,b\rangle\}, \\ & & \{\langle 1,a\rangle, \langle 1,b\rangle, \langle 2,a\rangle\}, \\ & & \{\langle 1,a\rangle, \langle 1,b\rangle, \langle 2,b\rangle\}, \\ & & \{\langle 1,a\rangle, \langle 2,a\rangle, \langle 2,b\rangle\}, \\ & & \{\langle 1,b\rangle, \langle 2,a\rangle, \langle 2,b\rangle\}, \\ & & \{\langle 1,a\rangle, \langle 1,b\rangle, \langle 2,a\rangle, \langle 2,b\rangle\}\} \end{array}$$

Look at the elements of this. Each is a possible graph/extension for a relation from S_1 to S_2. These include, e.g., the empty relation, the relation that maps 1 to a and nothing else, the relation that maps 1 to a and 1 to b and nothing else, the relation that maps 1 to a, 1 to b, 2 to a and nothing else, and the universal relation for these two sets. Clearly any relation from S_1 to S_2 will have a graph/extension that is one of the elements of this set.

Part (ii)

$$\mathit{card}(\wp(A_1 \times A_2))$$

e.g. if *card*$(A_1) = 3$ and *card*$(A_2) = 2$ then $A_1 \times A_2$ will have $3 \times 2 = 6$ elements, and the power set of a 6-element set has $2 \uparrow 6 = 64$ elements, so there are 64 relations.

Question 17 [*Hard; Optional, but you're encouraged to have a go at it*]

Part (i)

A unary function from natural numbers to natural numbers, e.g *double*

Part (ii)

A unary function from a set of elements from A to a natural number, e.g. *card*

Part (iii)

A binary function from a set of elements from A_1 and a set of elements from A_2 to a set of elements from A_1, e.g. set difference

Part (iv)

A binary function from a pair of sets (one drawn from A_1 and one from A_2) to a set of pairs (whose first co-ordinate is from A_1, and whose seocnd is from A_2), e.g. cartesian product

Part (v)

A unary function from a relation from set A_1 to set A_2 to a Boolean, e.g. *Total* (which says whether a relation is total or not)

Part (vi)

A unary function from a relation from set A_1 to set A_2 to a relation from set A_2 to set A_1, e.g. relational inverse

Part (vii)

A binary function from a pair of relations, one from A_1 to A_2, and one from A_2 to A_3, to a relation from A_1 to A_3, e.g. relational composition

Part (viii)

A binary function from a relation from A_1 to A_2 and a set drawn from A_1 to a set drawn from A_2, e.g. relational image

Part (ix)

A unary function that takes in a set drawn from A and returns a function from elements of A to Boolean, e.g. the 'operation' that converts a set to its characteristic function

Part (x)

A unary function that takes in a function from elements of A to Booleans and returns a set drawn from A, e.g. the 'operation' that converts a characteristic function of a set back to the original set

Question 18 [Optional: for the mathematically inclined]

The matrices we deal with have values of only 0 or 1 in their cells. Where I use matrix addition below, when adding the values in cells, we use the following: $0+0=0, 0+1=1; 1+0=1; 1+1=1$. (Note this last one: $1+1=1$; this means that addition here is more like a 'maximum' operation.) M^T denotes the transpose of matrix M. (In the transpose, M's rows and columns are interchanged.)

- For any relation R having matrix M, the matrix for R^{-1} is M^T.
- For relations R_1 and R_2 with matrices M_1 and M_2 then the matrix for $R_2 \circ R_1$ is given by $M_1 \times M_2$ where $\times$ here is matrix multiplication.
- A relation R on a set A of n elements will have an n by n matrix, M. Let I be the n by n identity matrix. Then
  - R is reflexive if and only if $I + M = M$
  - R is symmetric if and only if $M^T = M$
  - R is transitive if and only if $M + M^2 = M$
  - and
  $$I + M + M^2 + \ldots + M^{n-1}$$
  is the matrix for the reflexive transitive closure of relation R.