MATHS FOR INFORMATION PROCESSING

Problem Sheet 2

1

(i) Give *extensional* definitions for each of the following.

(a) The set of colleges of the University of York.

(b) The set of all positive even integers that are less than 3.

(c) $\{x \mid x/3 \in \mathbb{N} \text{ and } x < 15\}$

(d) $\{x \mid x/16 \in \mathbb{P} \text{ and } x < 15\}$

(ii) Give *intensional* definitions for each of the following. (You may use some English within the defining properties, but, on the whole, you should aim to keep this to a minimum.)

(a) The set of all positive even integers

(b) The set of all positive integers that are the square of some other positive integer

(c) $\{14, 12, 10, 8, 6, 4, 2\}$

(d) $\{3, 3, 4, 4, 5, 5, 6, 6\}$

In this question,

$$
\begin{aligned}
U &\triangleq \{x \mid x \in \mathbb{P} \text{ and } x \leq 10\} \\
S1 &\triangleq \{3, 6, 9\} \\
S2 &\triangleq \{2, 4, 6, 8, 10\} \\
S3 &\triangleq \{\emptyset, \{3\}, \{6\}, \{9\}, \{3, 6\}, \{3, 9\}, \{6, 9\}, \{3, 6, 9\}\}
\end{aligned}
$$

(i) Write down the cardinalities of each of the sets $U, S1, S2$ and $S3$.

(ii) Rewrite each of the following expressions, replacing the dots (...) with $\in, \subseteq$ and/or $\subset$ to make true statements. (Sometimes there will be more than one answer, in which case give them all. Sometimes there will be no answer, in which case write 'None'.)

Example: for $S1 \ldots U$ the answers are $S1 \subseteq U$ and $S1 \subset U$.

(a) $3 \ldots S1$

(b) $3 \ldots S3$

(c) $\{3\} \ldots S1$

(d) $\{3\} \ldots S3$

(e) $\{3, 9\} \ldots S1$

(f) $\{3, 9\} \ldots S3$

(g) $\{1, 3, 6, 9\} \ldots S3$

(h) $S2 \ldots U$

(i) $S1 \ldots S2$

(j) $S3 \ldots S3$

(k) $\emptyset \ldots U$

(l) $\emptyset \ldots S1$

(m) $\emptyset \ldots S3$

(n) $S3 \ldots U$

3

In this question, we are describing our Department's Autumn term courses for first year ITBML and for M.Sc.IP students.

$$
\begin{aligned}
U &\triangleq \{CSA, HI1, ICI, IPP, LB1, MIP, MIS\} \\
IT &\triangleq \{ICI, IPP, LB1, MIP\} \\
IP &\triangleq \{CSA, HI1, IPP, MIP, MIS\}
\end{aligned}
$$

(i) Give extensional definitions for each of the following sets.

(a) $IT \cup IP$

(b) $IT \cup IT$

(c) $IT \cap IP$

(d) $IP \cup U$

(e) $IP \cup \emptyset$

(f) $IP \cup \{\emptyset\}$

(g) $IT \cap \emptyset$

(h) $\{\emptyset\} \cap IT$

(i) $IT \cap U$

(j) IT'

(k) IP''

(l) $U \setminus IT$

(m) $IT \setminus IP$

(n) $\wp(IT)$

(o) $\wp(\{CSA\})$

(p) $\wp(\emptyset)$

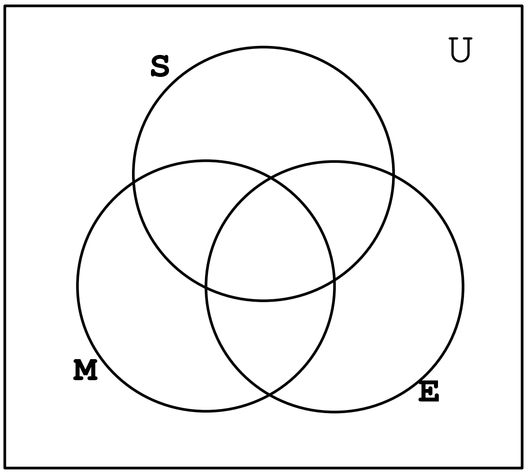
(q) $\wp(\wp(\emptyset))$

(ii) Write down *all* the possible partitions of IT. How many are there?

4

In a survey of 60 people, it was found that 25 read the Sun, 26 read the Mirror and 26 read the EXpress. Also, of these, 9 read both the Sun and Express, 11 read both the Sun and the Mirror, 8 read both the Mirror and Express, and 8 couldn't read at all (they just looked at the pictures).

(i) Let S, M and E denote the sets of Sun, Mirror and Express readers respectively. Fill in the correct numbers to show the cardinality of each of the 8 regions of the Venn diagram:



Use the following law to obtain your answer:

$$
\begin{aligned}
&card((A_1 \cup A_2) \cup A_3) = \\
&\quad card(A_1) + card(A_2) + card(A_3) \\
&\quad -card(A_1 \cap A_2) - card(A_1 \cap A_3) - card(A_2 \cap A_3) \\
&\quad +card((A_1 \cap A_2) \cap A_3)
\end{aligned}
$$

(ii) How many people read exactly one paper?

5 *Hard; Optional*

Using the laws of set algebra, some simple arithmetic and the following law,

$$\mathit{card}(A_1 \cup A_2) = (\mathit{card}(A_1) + \mathit{card}(A_2)) - \mathit{card}(A_1 \cap A_2)$$

prove that

$$\begin{aligned}
&\mathit{card}((A_1 \cup A_2) \cup A_3) = \\
&\quad \mathit{card}(A_1) + \mathit{card}(A_2) + \mathit{card}(A_3) \\
&\quad -\mathit{card}(A_1 \cap A_2) - \mathit{card}(A_1 \cap A_3) - \mathit{card}(A_2 \cap A_3) \\
&\quad +\mathit{card}((A_1 \cap A_2) \cap A_3)
\end{aligned}$$

6

(i) Prove the following two laws by, in each case, drawing Venn diagrams alongside each other and using suitable shading (which you should explain in some accompanying text).

(a) Distributivity of $\cap$ over $\cup$: for any sets SE_1, SE_2, SE_3,

$$SE_1 \cap (SE_2 \cup SE_3) = (SE_1 \cap SE_2) \cup (SE_1 \cap SE_3)$$

(b) One of De Morgan's laws: for any sets SE_1, SE_2,

$$(SE_1 \cap SE_2)' = SE_1' \cup SE_2'$$

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We gave no precedence or associativity conventions for the set operations. This wouldn't matter if, e.g., for any sets A_1, A_2 and A_3, $A_1 \cup (A_2 \cap A_3) = (A_1 \cup A_2) \cap A_3$ were true (i.e. if the evaluation order for these two operations didn't matter). But, this isn't true in general. Give a counterexample.

8

Set difference can be defined in terms of intersection and set complement as follows: for any sets SE_1, SE_2,

$$SE_1 \setminus SE_2 = SE_1 \cap SE_2'$$

(i) Prove this using the Venn diagram technique.

(ii) Replace the dots in the following by the simplest expressions that you think make them true statements. Then, prove the resulting statements algebraically, using the definition of set difference from above and the laws in the 'Examination Notes'.

(a) For any set A, $A \setminus \emptyset = \ldots$

(b) For any set A, $A \setminus U = \ldots$

(c) For any set A, $A \setminus A = \ldots$

(iii) Very occasionally books present De Morgan's laws using set difference instead of set complement, as follows: for any sets A_1, A_2, A_3,

$$A_1 \setminus (A_2 \cup A_3) = (A_1 \setminus A_2) \cap (A_1 \setminus A_3)$$

$$A_1 \setminus (A_2 \cap A_3) = (A_1 \setminus A_2) \cup (A_1 \setminus A_3)$$

Prove both of these laws by algebraic manipulation (*not* by using Venn diagrams). Make use of the definition of set difference in terms of intersection and complement from above and any laws given in the 'Examination Notes' that you need.

(iv) In fact, union can also be defined in terms of intersection and complement as follows: for any sets A_1, A_2,

$$A_1 \cup A_2 = (A_1' \cap A_2')'$$

Prove this algebraically using the laws given in the 'Examination notes'.

(By the way, the various parts of this questions have shown that only two of the four operations are really needed (only two are 'primitive'); the other two can be defined in terms of these two.)

Model Answers

MATHS FOR INFORMATION PROCESSING

Problem Sheet 2

Question 1

Part (i)

Option (a)

{Alcuin, Derwent, Goodricke, James, Langwith, Vanbrugh, Wentworth}

Option (b)

$\{2\}$

Option (c)

$\{0, 3, 6, 9, 12\}$

Option (d)

$\emptyset$

Part (ii)

Option (a)

$\{x \mid x/2 \in \mathbb{P}\}$

Option (b)

$\{x \mid x \in \mathbb{P}$ and there is some $y \in \mathbb{P}$ such that $y \uparrow 2 = x\}$

Option (c)

$\{x \mid x/2 \in \mathbb{P}$ and $x \leq 14\}$

Option (d)

$\{x \mid x \in \mathbb{N}$ and $x \geq 3$ and $x \leq 6\}$

Question 2

Part (i)

$$\begin{aligned} \textit{card}(U) &= 10 \\ \textit{card}(S1) &= 3 \\ \textit{card}(S2) &= 5 \\ \textit{card}(S3) &= 8 \end{aligned}$$

Part (ii)
Option (a)
$3 \in S1$
Option (b)
None
Option (c)
$\{3\} \subseteq S1, \{3\} \subset S1$
Option (d)
$\{3\} \in S3$
Option (e)
$\{3, 9\} \subseteq S1, \{3, 9\} \subset S1$
Option (f)
$\{3, 9\} \in S3$
Option (g)
None
Option (h)
$S2 \subseteq U; S2 \subset U$
Option (i)
None
Option (j)
$S3 \subseteq S3$
Option (k)
$\emptyset \subseteq U; \emptyset \subset U$
Option (l)
$\emptyset \subseteq S1, \emptyset \subset S1$
Option (m)
$\emptyset \in S3, \emptyset \subseteq S3, \emptyset \subset S3$
Option (n)
None

Question 3

Part (i)
Option (a)
{CSA, HI1, ICI, IPP, LB1, MIP, MIS}

Option (b)

{ICI, IPP, LB1, MIP}

Option (c)

{IPP, MIP}

Option (d)

{CSA, HI1, ICI, IPP, LB1, MIP, MIS}

Option (e)

{CSA, HI1, IPP, MIP, MIS}

Option (f)

{CSA, HI1, IPP, MIP, MIS, ∅}

Option (g)

∅

Option (h)

∅

Option (i)

{ICI, IPP, LB1, MIP}

Option (j)

{CSA, HI1, MIS}

Option (k)

{CSA, HI1, IPP, MIP, MIS}

Option (l)

{CSA, HI1, MIS}

Option (m)

{ICI, LB1}

Option (n)

{∅,			
{ICI},	*{IPP},*	*{LB1},*	*{MIP},*
{ICI, IPP},	*{ICI, LB1},*	*{ICI, MIP},*	
{IPP, LB1},	*{IPP, MIP},*	*{LB1, MIP},*	
{ICI, IPP, LB1},	*{ICI, IPP, MIP},*	*{ICI, LB1, MIP},*	*{IPP, LB1, MIP},*
{ICI, IPP, LB1, MIP}}			

Option (o)

{∅, {CSA}}

Option (p)

{∅}

Option (q)

$\{\emptyset, \{\emptyset\}\}$

Part (ii)

There are 15 different partitions.

{{ICI}, {IPP}, {LB1}, {MIP}}

{{ICI, IPP}, {LB1, MIP}} *{{ICI, LB1}, {IPP, MIP}}* *{{ICI, MIP}, {LB1, IPP}}*

{{ICI}, {IPP}, {LB1, MIP}} *{{ICI}, {LB1}, {IPP, MIP}}* *{{ICI}, {MIP}, {LB1, IPP}}*
{{ICI, IPP}, {LB1}, {MIP}} *{{ICI, LB1}, {IPP}, {MIP}}* *{{ICI, MIP}, {LB1}, {IPP}}*

{{ICI}, {IPP, LB1, MIP}} *{{IPP}, {ICI, LB1, MIP}}*
{{LB1}, {ICI, IPP, MIP}} *{{MIP}, {ICI, IPP, LB1}}*

{{ICI, IPP, LB1, MIP}}

Question 4

Part (i)

8 can't read, so this is the number of people who fall outside of the three circles.

The total no. of people, 60, less this 8, $= 52$, is the total no. who fall within the three circles.

Plugging values into the law from the previous question:

$$52 = 25 + 26 + 26 - 11 - 9 - 8 + \mathit{card}((S \cap M) \cap E)$$

Hence, $\mathit{card}((S \cap M) \cap E) = 3$.

$9 - 3 = 6$ read the Sun and Express but not all three.

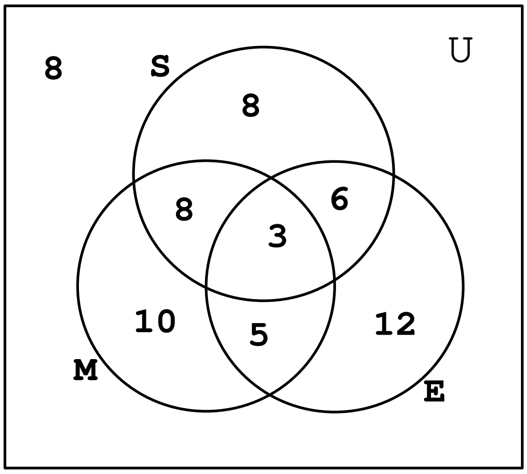
$11 - 3 = 8$ read the Sun and the Mirror but not all three.

$8 - 3 = 5$ read the Mirror and Express but not all three.

$25 - 6 - 8 - 3 = 8$ read the Sun only.

$26-8-5-3=10$ read the Mirror only.

$26-6-5-3=12$ read Express only.



Part (ii)

$8+10+12=30.$

Question 5 *Hard; Optional*

$$\begin{array}{rcll} \text{LHS} & = & \mathit{card}((A_1 \cup A_2) \cup A_3) & \\ & = & (\mathit{card}(A_1 \cup A_2) + \mathit{card}(A_3)) & \\ & & -\mathit{card}((A_1 \cup A_2) \cap A_3) & \text{(cardinality of union)} \\ & = & ((\mathit{card}(A_1) + \mathit{card}(A_2)) & \\ & & -\mathit{card}(A_1 \cap A_2) + \mathit{card}(A_3)) & \\ & & -\mathit{card}((A_1 \cup A_2) \cap A_3) & \text{(cardinality of union)} \end{array}$$

Let's work on the last part of this, $\mathit{card}((A_1 \cup A_2) \cap A_3)$:

$$
\begin{aligned}
\mathit{card}((A_1 \cup A_2) \cap A_3) &= \mathit{card}(A_3 \cap (A_1 \cup A_2)) && \text{(commutativity of } \cap\text{)} \\
&= \mathit{card}((A_3 \cap A_1) \cup (A_3 \cap A_2)) && \text{(distributivity of } \cap \text{ over } \cup\text{)} \\
&= \mathit{card}((A_1 \cap A_3) \cup (A_3 \cap A_2)) && \text{(commutativity of } \cap\text{)} \\
&= \mathit{card}((A_1 \cap A_3) \cup (A_2 \cap A_3)) && \text{(commutativity of } \cap\text{)} \\
&= \mathit{card}(A_1 \cap A_3) + \mathit{card}(A_2 \cap A_3) && \\
&\quad -\mathit{card}((A_1 \cap A_3) \cap (A_2 \cap A_3)) && \text{(cardinality of union)} \\
&= \mathit{card}(A_1 \cap A_3) + \mathit{card}(A_2 \cap A_3) && \\
&\quad -\mathit{card}((A_1 \cap A_3) \cap (A_3 \cap A_2)) && \text{(commutativity of } \cap\text{)} \\
&= \mathit{card}(A_1 \cap A_3) + \mathit{card}(A_2 \cap A_3) && \\
&\quad -\mathit{card}(A_1 \cap (A_3 \cap (A_3 \cap A_2))) && \text{(associativity of } \cap\text{)} \\
&= \mathit{card}(A_1 \cap A_3) + \mathit{card}(A_2 \cap A_3) && \\
&\quad -\mathit{card}(A_1 \cap ((A_3 \cap A_3) \cap A_2)) && \text{(associativity of } \cap\text{)} \\
&= \mathit{card}(A_1 \cap A_3) + \mathit{card}(A_2 \cap A_3) && \\
&\quad -\mathit{card}(A_1 \cap (A_3 \cap A_2)) && \text{(idempotence)} \\
&= \mathit{card}(A_1 \cap A_3) + \mathit{card}(A_2 \cap A_3) && \\
&\quad -\mathit{card}(A_1 \cap A_2 \cap A_3) && \text{(commutativity of } \cap \\
& && \text{and dropping a pair of} \\
& && \text{parentheses, as licensed} \\
& && \text{by associativity of } \cap\text{)}
\end{aligned}
$$

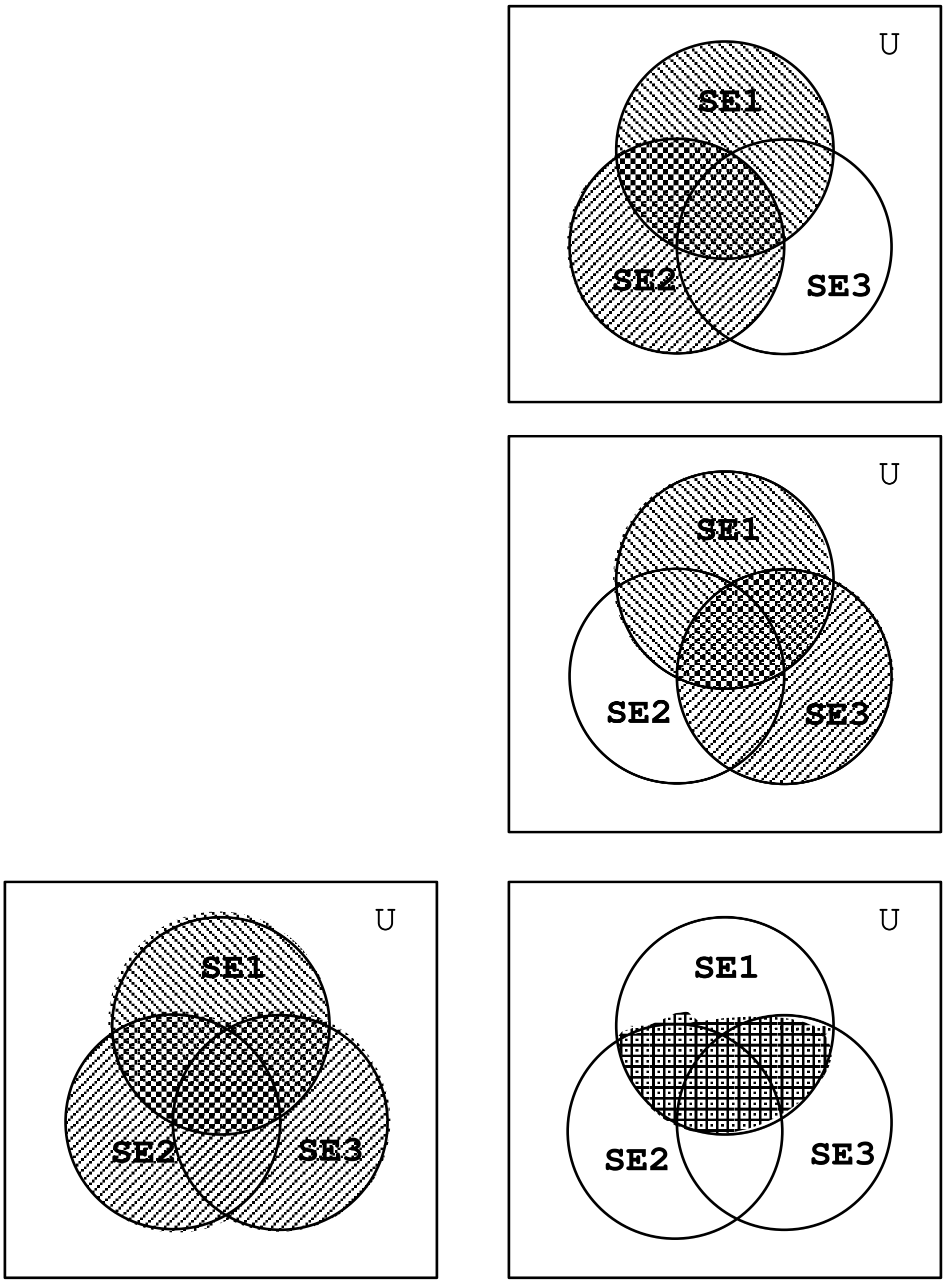
Now plug this back into what we had earlier:

$$
\begin{aligned}
\text{LHS} &= ((\mathit{card}(A_1) + \mathit{card}(A_2)) \\
&\quad -\mathit{card}(A_1 \cap A_2) + \mathit{card}(A_3)) \\
&\quad -(\mathit{card}(A_1 \cap A_3) + \mathit{card}(A_2 \cap A_3) \\
&\quad -\mathit{card}(A_1 \cap A_2 \cap A_3)) \\
&= ((\mathit{card}(A_1) + \mathit{card}(A_2)) \\
&\quad -\mathit{card}(A_1 \cap A_2) + \mathit{card}(A_3)) \\
&\quad -\mathit{card}(A_1 \cap A_3) - \mathit{card}(A_2 \cap A_3) \\
&\quad +\mathit{card}(A_1 \cap A_2 \cap A_3) && \text{(simple arithmetic)} \\
&= \text{RHS}
\end{aligned}
$$

Question 6

Part (i)

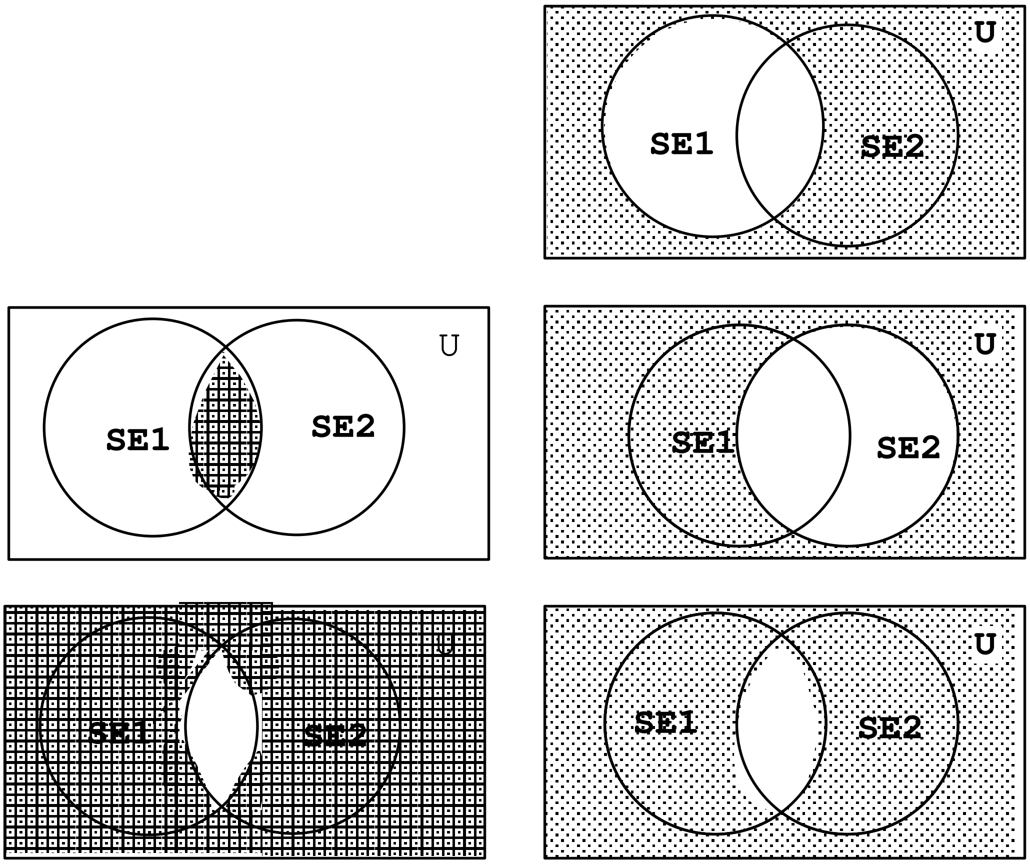
Option (a)



LHS diagram uses ///// to show $SE_2 \cup SE_3$, then \\\\\ to show SE_1 so the dark area in the middle is $SE_1 \cap (SE_2 \cup SE_3)$.

The RHS diagrams show firstly $(SE_1 \cap SE_2)$ (dark area), secondly $(SE_1 \cap SE_3)$ (dark area) and thirdly the union of these two dark areas. This is the same as the dark shaded area in LHS diagram, so the law is proved.

Option (b)



The LHS diagrams show firstly $(SE_1 \cap SE_2)$ as a shaded area, and secondly the complement of this as a shaded area.

The RHS diagrams show firstly SE_1' as a shaded area, secondly SE_2' as a shaded area, and thirdly the union of these two shaded areas as a shaded area. This shaded area is the same as the shaded area in the LHS diagram, so the law is proved.

Question 7

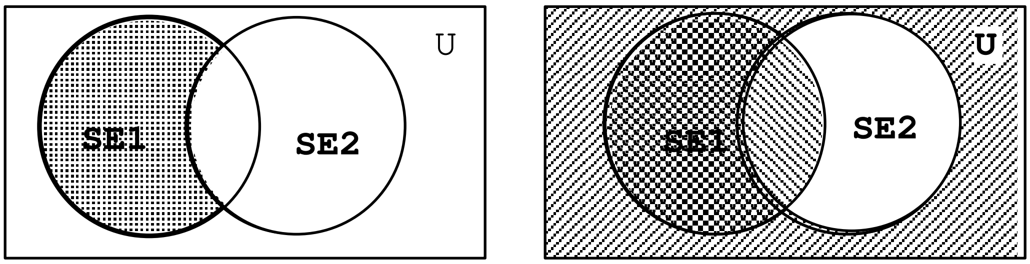
Take A_1 to be $\{1, 2\}$, A_2 to be $\{2, 3\}$ and A_3 to be $\{3, 4\}$. Then $A_1 \cup (A_2 \cap A_3) =$

$\{1,2\} \cup (\{2,3\} \cap \{3,4\}) = \{1,2\} \cup \{3\} = \{1,2,3\}$. But $(A_1 \cup A_2) \cap A_3 = (\{1,2\} \cup \{2,3\}) \cap \{3,4\} = \{1,2,3\} \cap \{3,4\} = \{3\}$.

Since we have given no precedence and associativity rules, we must use parentheses liberally in our expressions of set theory.

Question 8

Part (i)



The LHS diagram shows $SE_1 \setminus SE_2$ as a shaded area.

The RHS diagram shows SE_2' shaded using ///// and SE_1 shaded using \\\\\\. Where these shadings overlap (darkest area) is $SE_1 \cap SE_2'$ and this is the same as the shaded area in the LHS diagram, so the law is proved.

Part (ii)
Option (a)

For any set A, $A \setminus \emptyset = A$

$$
\begin{aligned}
\text{LHS} &= A \setminus \emptyset \\
&= A \cap \emptyset' && \text{(defn of set difference)} \\
&= A \cap U && \text{(complement law)} \\
&= U \cap A && \text{(commutativity of } \cap\text{)} \\
&= A && \text{(identity)} \\
&= \text{RHS}
\end{aligned}
$$

Option (b)

For any set A, $A \setminus U = \emptyset$

$$
\begin{aligned}
\text{LHS} &= A \setminus U \\
&= A \cap U' && \text{(defn of set difference)} \\
&= A \cap \emptyset && \text{(complement law)} \\
&= \emptyset \cap A && \text{(commutativity of } \cap\text{)} \\
&= \emptyset && \text{(identity)} \\
&= \text{RHS}
\end{aligned}
$$

Option (c)

For any set A, $A \setminus A = \emptyset$

$$\begin{aligned}
\text{LHS} &= A \setminus A \\
&= A \cap A' && \text{(defn of set difference)} \\
&= \emptyset && \text{(complement law)} \\
&= \text{RHS}
\end{aligned}$$

Part (iii)

To show

$$A_1 \setminus (A_2 \cup A_3) = (A_1 \setminus A_2) \cap (A_1 \setminus A_3)$$

$$\begin{aligned}
\text{LHS} &= A_1 \setminus (A_2 \cup A_3) \\
&= A_1 \cap (A_2 \cup A_3)' && \text{(defn of set difference)} \\
&= A_1 \cap (A_2' \cap A_3') && \text{(original De Morgan's laws)} \\
&= (A_1 \cap A_1) \cap (A_2' \cap A_3') && \text{(idempotence of } \cap) \\
&= A_1 \cap (A_1 \cap (A_2' \cap A_3')) && \text{(associativity of } \cap) \\
&= A_1 \cap ((A_1 \cap A_2') \cap A_3') && \text{(associativity of } \cap) \\
&= ((A_1 \cap A_2') \cap A_3') \cap A_1 && \text{(commutativity of } \cap) \\
&= (A_1 \cap A_2') \cap (A_3' \cap A_1) && \text{(associativity of } \cap) \\
&= (A_1 \cap A_2') \cap (A_1 \cap A_3') && \text{(commutativity of } \cap) \\
&= (A_1 \setminus A_2) \cap (A_1 \cap A_3') && \text{(defn of set difference)} \\
&= (A_1 \setminus A_2) \cap (A_1 \setminus A_3) && \text{(defn of set difference)} \\
&= \text{RHS}
\end{aligned}$$

The other is similar. For variety, I'll prove this one by working on the RHS. To show

$$A_1 \setminus (A_2 \cap A_3) = (A_1 \setminus A_2) \cup (A_1 \setminus A_3)$$

$$\begin{aligned}
\text{RHS} &= (A_1 \setminus A_2) \cup (A_1 \setminus A_3) \\
&= (A_1 \cap A_2') \cup (A_1 \cap A_3') && \text{(two uses of the defn of set difference)} \\
&= A_1 \cap (A_2' \cup A_3') && \text{(distributivity of } \cap \text{ over } \cup) \\
&= A_1 \cap (A_2 \cap A_3)' && \text{(original De Morgan's laws)} \\
&= A_1 \setminus (A2 \cap A_3) && \text{(defn of set difference)} \\
&= \text{LHS}
\end{aligned}$$

Part (iv)

In this one, I'll work on the RHS and show it equals the LHS.

$$
\begin{aligned}
\text{RHS} &= (A_1' \cap A_2')' \\
&= (A_1'' \cup A_2'') && \text{(De Morgan's law)} \\
&= (A_1 \cup A_2'') && \text{(involution)} \\
&= (A_1 \cup A_2) && \text{(involution)} \\
&= \text{LHS}
\end{aligned}
$$