MATHS FOR INFORMATION PROCESSING
Problem Sheet 1

1

Say which of the following expressions are *syntactically well-formed* arithmetic expressions and which are not. (Hint: consult the page of your notes on which the formulation of arithmetic expressions is defined by four numbered points and use those four points quite literally. The more literal-minded you are, then, for the most part, the less likely you are to be fooled by those that I have included to fool you!)

(i) $(3+2) \uparrow 0$

(ii) $(((3)+(2)) \uparrow (0))$

(iii) $\sin(\cos(\tan(45)))$

(iv) $3 * /2$

(v) $3!!$

(vi) $(((4) * (3) - (1))$

(vii) $3\ \mathit{cheese}\ 4$

(viii) $\sin(90, 10)$

(ix) $\mathit{onion} + 2 * \mathit{onion}$

(x) $(3\ \mathit{salt}\ 4) + (\ \mathit{salt}\ \ * 2)$

(xi) $12/(3-3)$

(xii) $(2 > 1) = (3 \geq 2)$

(xiii) $2 > 1 = 3 \geq 2$

(xiv) $3(+)2$

(xv) $+(2, 3)$

(xvi) $3 \leq n < 7$

(xvii) $\mathit{vinegar}\ \triangleq 18 * 2$

(xviii) $(2 + 3) = (6 - 1)$

(xix) $2 + (3 = 6) - 1$

(xx) $0 - 0$

(xxi) $12/3/96$

(xxii) $3 \uparrow 2 \uparrow$

(xxiii) $([(3 \uparrow 2)/12] - 4)$

(xxiv) $(10 \div 2) - 3$

(xxv) $(\sin + \cos)(90)$

(xxvi) $3(10 + 12)$

2

For each of the expressions in question 1 that you said were syntactically well-formed, now say whether they are semantically well-formed, i.e. will they cause problems in evaluation? (Hint: *among several things to look for*, one is to see whether, using the given rules for precedence and associativity of the operators, you would find yourself trying to apply an operation to the wrong *type* of object.)

3

Rewrite the following expressions, inserting additional parentheses to make explicit the way that the operator precedence and associativity conventions given in the notes would affect the evaluation order.

(i) $3 + 4 * 5$

(ii) $3 * 4 + 5$

(iii) $3 + 4 + 5$

(iv) $3/4/5$

(v) $3 \uparrow 4 \uparrow 5$

(vi) $3 \uparrow 4 \uparrow 5 * 6$

(vii) $2 + 3 = 6 - 1$

(viii) $3 - 4 - 5 - 6 * 7 \uparrow 8$

4

In BizarreArithmetic (a version of arithmetic which I've just made up), the only operators are those whose precedence and associativity conventions are shown below.

Precedence	
$+, \uparrow$	highest
$/$	
$*$	
$-$	
$=$	lowest

Associate to the left	Associate to the right
$+, \uparrow, *, =$	$/, -$

Say whether the following expressions evaluate to true or false according to these bizarre conventions, i.e. does the left hand side evaluate to the same number as the right hand side?

(i) $3 + 4 * 5 = 3 + (4 * 5)$

(ii) $5 - 4 - 3 - 2 = 2$

(iii) $10/(12/3)/2 = 5 \uparrow 0 \uparrow 3 * 5$

5

In MoreBizarreArithmetic (another of my inventions), the only operators are those whose precedence and associativity conventions are shown below.

Precedence	
$\uparrow$	highest
$*, /$	
$+, -$	lowest

Associate to the left	Associate to the right
$*, /, +$	$\uparrow, -$

Explain using an example why this set of conventions is problematic.

6

(i) Give definitions (using $\triangleq$) of the following operators.

(a) A unary prefix operator *square* that squares its operand.

(b) A unary prefix operator *quad*, defined using *square* rather than $\uparrow$, that raises its operand to the fourth power.

(ii) Assume for the next two questions that you have been given a definition of a binary infix operator *mod* such that for numbers x_1 and x_2, x_1 *mod* x_2 returns the remainder when x_1 is divided by x_2. Use *mod* to define the following operators.

(a) A unary prefix operator *even* that returns true if its operand is an even number and false otherwise.

(b) A binary prefix operator *intdiv* that returns the same as /, but gives only an integer result, e.g. $14/4 = 3.5$ so 14 *intdiv* 4 = 3.

(iii) Firstly, write an *expression* that computes the cost of having a room fitted with wall-to-wall carpeting: the room is rectangular, its length being 5 metres and its width being 3 metres; the carpet costs £10 per square metre, and there is a fixed service charge of £50.

Secondly, define a 4-ary prefix operator that takes in the length and width of a room, the cost of a carpet per square metre and a fixed service charge and returns the cost of the installation. accordingly.

7

For each occurrence of a name below (other than the 'standard' arithmetic operators and the numerals), show whether the occurrence is a binding occurrence (in which case, write a unique number alongside it). If it is a bound applied occurrence, then write the letter 'B' and the reference number of the corresponding binding occurrence alongside it. If it is an applied occurrence but is free, then write the letter 'F' alongside it.

In this question,

$$O1(x) \triangleq x + 1$$

$$O2(x, y) \triangleq (O1(x) \uparrow 2) + (O1(y) \uparrow 2)$$

$$O3(x) \triangleq O2(z, z)$$

Expressions

$$O1(O3(4))$$

$$O4(O1(5))$$

8

Give counterexamples that show that

(i) / is not commutative

(ii) / is not associative

(iii) / is not distributive over $*$

(iv) / is not distributive over $+$

9

Using the laws given in the notes, prove the following:

(i) For all E_1, E_2, E_3,

$$(E_1 + E_2) * E_3 = (E_1 * E_3) + (E_2 * E_3)$$

(ii) For all E_1, E_2, E_3,

$$(E_1 * E_2) + (E_2 * E_3) = E_2 * (E_1 + E_3)$$

(iii) For all E,

$$(E * 1) + (E * 0) = E$$

Model Answers

Maths for Information Processing

Problem Sheet 1

Question 1

Part (i)

$(3+2) \uparrow 0$ is.

Part (ii)

$(((3)+(2)) \uparrow (0))$ is.

Part (iii)

$\sin(\cos(\tan(45)))$ is.

Part (iv)

$3 * /2$ is not: two infix operators abut.

Part (v)

$3!!$ is: it is the factorial of the factorial of 3, i.e. $(3!)!$.

Part (vi)

$(((4) * (3) - (1))$ is not: a parenthesis is missing.

Part (vii)

3 *cheese* 4 is: while the name *cheese* is free (not bound), if we take it to be a binary infix operator then the expression is well-formed.

Part (viii)

$\sin(90, 10)$ is not: $\sin$ is a unary operator.

Part (ix)

onion $+ 2 *$ *onion* is: as with *cheese* in part (vii), *onion* is free, but if we take it in this case to be the name of a number then the expression is well-formed.

Part (x)

$(3$ *salt* $4) + ($ *salt* $* 2)$ is not: *salt* is free and we cannot take it to be both a binary infix operator (for well-formedness of $(3$ *salt* $4)$) and the name of a number (for well-formedness of $($*salt* $* 2)$).

Part (xi)

$12/(3-3)$ is.

Part (xii)

$(2>1)=(3\geq 2)$ is: each operator is binary and infix, and, sure enough, each has two operands either side of it.

Part (xiii)

$2>1=3\geq 2$ is: for the same reasons as for (xii).

Part (xiv)

$3(+)2$ is not: parentheses must enclose *expressions*. An expression is either a numeral (or other name for a number) or an operator with its operands; an operator on its own is not an expression.

Part (xv)

$+(2,3)$ is not: $+$ is defined to be infix not prefix.

Part (xvi)

$3\leq n<7$ is: the free occurrence of n is not a problem (see, e.g. (ix)). Other than that, the expression is a bit like, e.g., $2+3+4$, which is fine, so this too is fine (but see the answer to Question 2 below).

Part (xvii)

vinegar $\triangleq 18*2$ is not: of course, there is no ill-formedness in this sequence of symbols *when looked at as a definition of a new name*. However, the task here is to say whether sequences of symbols are well-formed *arithmetic expressions*, not definitions. Since this is not an expression, it is not a well-formed expression. The giveaway sign that this is not an arithmetic expression is the presence of the symbol $\triangleq$, which is a symbol from our metalanguage.

Part (xviii)

$(2+3)=(6-1)$ is.

Part (xix)

$2+(3=6)-1$ is: each operator is binary and infix, and, sure enough, each has two operands either side of it.

Part (xx)

$0-0$ is.

Part (xxi)

$12/3/96$ is: the expression is one involving two divisions $((12/3)/96)$. It is not a date!

Part (xxii)

$3 \uparrow 2 \uparrow$ is not: the second use of $\uparrow$ has only one operand.

Part (xxiii)

$([(3 \uparrow 2)/12] - 4)$ is not: we have legitimised parentheses, not square brackets. (Of course, in practice, any sensible use of 'punctuation' to aid legibility, as here, is not to be frowned upon.)

Part (xxiv)

$(10 \div 2) - 3$ is: we have been using / for division, so this makes the symbol $\div$ free. Hence, this example is like (vii).

Part (xxv)

$(\sin + \cos)(90)$ is not: $\sin$ and $\cos$ are unary operators and so need an operand each; perhaps the intention of whoever wrote this was that they should 'share' the 90, but this is not allowed.

Part (xxvi)

$3(10 + 12)$ is not: it is common in maths to drop the symbol $*$ and use juxtaposition where possible (so the expression would be shorthand for $3 * (10 + 12)$); but this is not allowed according to our four rules for well-formed expressions.

Question 2

(i) $(3 + 2) \uparrow 0$: no problem.

(ii) $(((3) + (2)) \uparrow (0))$: no problem.

(iii) $\sin(\cos(\tan(45)))$: no problem.

(v) $3!!$: no problem.

(vii) 3 *cheese* 4: we will not be able to evaluate this without a binding for the operator *cheese*.

(ix) *onion* $+ 2 *$ *onion*: similarly, here we need a binding for *onion*.

(xi) $12/(3-3)$: most computers and some calculators will have problems with this because a division by zero arises, and division of any number by zero is undefined.

(xii) $(2>1)=(3\geq 2)$: no problem. $(2>1)$ evaluates to true; $(3\geq 2)$ evaluates to true; true = true, so the whole expression is true.

(xiii) $2>1=3\geq 2$: this one is problematic. The operators $>, =$ and $\geq$ all have the same precedence but they associate to the left, so this expression is evaluated the same as $((2>1)=3)\geq 2$. The expression $(2>1)$ evaluates to true; then we ask is true equal to the number 3, which is false. Finally, we ask whether false is greater than or equal to the number 2. Now we definitely have an expression with mismatched types. So this is semantically problematic.

(xvi) $3\leq n<7$: this is problematic. Firstly, we need a value for n. Secondly, $3\leq n$ will return a Boolean and we cannot then ask whether this is less than the number 7 (mismatched types). Note that this kind of expression does crop up in maths books, but, where it does, it is almost invariably being used as a shorthand for '$3\leq n$ and $n<7$', i.e. the sticking together of multiple statements.

(xviii) $(2+3)=(6-1)$: no problem.

(xix) $2+(3=6)-1$: $(3=6)$ is false, and we cannot then add 2 to false.

(xx) $0-0$: no problem.

(xxi) $12/3/96$: no problem.

(xxiv) $(10\div 2)-3$: there is the problem of knowing what operation $\div$ is bound to.

Question 3

Part (i)

$3+(4*5)$

Part (ii)

$(3*4)+5$

Part (iii)

$(3 + 4) + 5$

Part (iv)

$(3/4)/5$

Part (v)

$3 \uparrow (4 \uparrow 5)$

Part (vi)

$(3 \uparrow (4 \uparrow 5)) * 6$

Part (vii)

$(2 + 3) = (6 - 1)$

Part (viii)

$((3 - 4) - 5) - (6 * (7 \uparrow 8))$

Question 4

Part (i)

$3 + 4 * 5 = 3 + (4 * 5)$ evaluates to false.

Part (ii)

$5 - 4 - 3 - 2 = 2$ evaluates to true.

Part (iii)

$10/(12/3)/2 = 5 \uparrow 0 \uparrow 3 * 5$ evaluates to true.

Question 5

Consider an expression such as $4+3-2$. All the operators in this expression have the same precedence, so we resort to associativity to determine which operation to carry out first. However, their associativities conflict: that of $-$ suggests that we evaluate $4 + (3 - 2)$ but that of $+$ suggests that we evaluate $(4 + 3) - 2$. The expression remains ambiguous. Operations of the same precedence (such as $+$ and $-$) shouldn't have different associativities.

Question 6

Part (i)
Option (a)

$$\mathit{square}(x) \triangleq x * x$$

(Obviously

$$\mathit{square}(x) \triangleq x \uparrow 2$$

would do just as well.)

Option (b)

$$\mathit{quad}(x) \triangleq \mathit{square}(\mathit{square}(x))$$

Part (ii)
Option (a)

$$\mathit{even}(x) \triangleq (x\ \mathit{mod}\ 2) = 0$$

Option (b)

$$\mathit{intdiv}(x, y) \triangleq (x - (x\ \mathit{mod}\ y))/y$$

Part (iii)

The expression is:

$$((5 * 3) * 10) + 50$$

The definition is

$$\begin{aligned}&\mathit{cost}(\mathit{length}, \mathit{width}, \mathit{costpersqm}, \mathit{fixedcost}) \triangleq \\ &\quad ((\mathit{length} * \mathit{width}) * \mathit{costpersqm}) + \mathit{fixedcost}\end{aligned}$$

Question 7

Definitions

$$\boxed{\text{O1}}_1(\boxed{\text{x}}_2) \triangleq \boxed{\text{x}}_{B2} + 1$$

$$\boxed{\text{O2}}_3(\boxed{\text{x}}_4, \boxed{\text{y}}_5) \triangleq (\boxed{\text{O1}}_{B1}(\boxed{\text{x}}_{B4}) \uparrow 2) + (\boxed{\text{O1}}_{B1}(\boxed{\text{y}}_{B5}) \uparrow 2)$$

$$\boxed{\text{O3}}_6(\boxed{\text{x}}_7) \triangleq \boxed{\text{O2}}_{B3}(\boxed{\text{z}}_F, \boxed{\text{z}}_F)$$

Expressions

$$\boxed{\text{O1}}_{B1}(\boxed{\text{O3}}_{B6}(4))$$

$$\boxed{\text{O4}}_F(\boxed{\text{O1}}_{B1}(5))$$

Question 8

Part (i)

/ is not commutative: e.g. $4/2 = 2$ but $2/4 = 0.5$

Part (ii)

/ is not associative: e.g. $(8/4)/2 = 1$ but $8/(4/2) = 4$

Part (iii)

/ is not distributive over $*$: e.g. $4/(4 * 2) = 0.5$ but $(4 * 4)/(4 * 2) = 2$

Part (iv)

/ is not distributive over $+$: e.g. $3/(4 + 2) = 0.5$ but $(3/4) + (3/2) = 2.25$

Question 9

Part (i)

$$\begin{aligned}
\text{LHS} &= (E_1 + E_2) * E_3 \\
&= E_3 * (E_1 + E_2) && \text{(commutativity of } *) \\
&= (E_3 * E_1) + (E_3 * E_2) && \text{(distributivity of } * \text{ over } +) \\
&= (E_1 * E_3) + (E_3 * E_2) && \text{(commutativity of } *) \\
&= (E_1 * E_3) + (E_2 * E_3) && \text{(commutativity of } *) \\
&= \text{RHS}
\end{aligned}$$

Part (ii)

$$\begin{aligned}
\text{LHS} &= (E_1 * E_2) + (E_2 * E_3) \\
&= (E_2 * E_1) + (E_2 * E_3) && \text{(commutativity of } *\text{)} \\
&= E_2 * (E_1 + E_3) && \text{(distributivity of } * \text{ over } +\text{)} \\
&= \text{RHS}
\end{aligned}$$

Part (iii)

$$\begin{aligned}
\text{LHS} &= (E * 1) + (E * 0) \\
&= E * (1 + 0) && \text{(distributivity of of } * \text{ over } +\text{)} \\
&= E * (0 + 1) && \text{(commutativity of } +\text{)} \\
&= E * 1 && \text{(identity)} \\
&= 1 * E && \text{(commutativity of } *\text{)} \\
&= E && \text{(identity)} \\
&= \text{RHS}
\end{aligned}$$