061311R

THE UNIVERSITY of York

B.Sc. and B.A. Examinations 1996

DEPARTMENT OF COMPUTER SCIENCE

Part I Resit

Mathematics for Information Processing

Time allowed: **One-and-a-half hours**

Silent non-programmable calculators may be used.
A set of 'Examination Notes' is attached.
Candidates should answer **all** questions from this paper.

In this paper the following set names are used throughout:

$$\begin{aligned} Suits &\triangleq \{Clubs, Diamonds, Hearts, Spades\} \\ Values &\triangleq \{A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K\} \\ Pictures &\triangleq \{J, Q, K\} \\ Players &\triangleq \{Ann, Ben, Col\} \end{aligned}$$

1 (10 marks)

(i) [3 marks] Give *extensional* definitions of each of the following sets:

(a) [$\frac{1}{2}$ marks] $Values \cap Pictures$

(b) [$\frac{1}{2}$ marks] $Values \setminus Pictures$

(c) [$\frac{1}{2}$ marks] $Pictures \setminus Values$

(d) [$\frac{1}{2}$ marks] $\wp(\{Q, K\})$

(e) [1 marks] $Suits \times (Pictures \setminus \{J\})$

(ii) [2 marks] State whether the following are true or false:

(a) [$\frac{1}{2}$ marks] $Pictures \subset Values$

(b) [$\frac{1}{2}$ marks] $\{Clubs\} \in Suits$

(c) [$\frac{1}{2}$ marks] $Pictures \in \wp(Values)$

(d) [$\frac{1}{2}$ marks] $Pictures \subseteq \wp(Values)$

(iii) [5 marks] Prove the following algebraically using the laws given in the 'Examination Notes'. You **must** annotate each step of your proof with the algebraic law that is used in that step.

For any sets A_1 and A_2,

$$A_1 \cup A_2 = A_2 \cup ((A_2' \cup (A_1 \cup A_2)) \cap A_1)$$

2 (20 marks)

(i) [12 marks]

(a) [3 marks] Cut is a relation from $Players$ to $Values$. It maps a player to the value of the card that s/he turns up in a cut of a deck of cards:

$$\begin{array}{rcl} Cut & : & Players \leftrightarrow Values \\ Gr(Cut) & \triangleq & \{\langle \mathit{Ann}, 3\rangle, \langle \mathit{Ben}, J\rangle, \langle \mathit{Col}, 3\rangle\} \end{array}$$

- Is Cut *total* or *partial*?
- Is it *surjective* or not?
- Is it *functional* or not?
- Is it *injective* or not?
- Is it *one-to-one*, *one-to-many*, *many-to-one* or *many-to-many*?
- Is it a *bijection*?

(b) [6 marks] Write expressions involving the Cut relation that, when evaluated, would compute:

- the set of cards turned up in a cut by Ann and Col [1 mark];
- the set of players who have turned up a Jack (J) in a cut [2 marks];
- the relation on $Players$ whose graph maps a player to a player who has turned up a card of the same value in a cut [3 marks].

(c) [3 marks] The $Deal$ relation is *incompletely* specified below:

$$\begin{array}{rcl} Deal & : & Players \leftrightarrow \ldots \\ Gr(Deal) & \triangleq & \{\langle Ann, \{\langle 2, Hearts\rangle, \langle J, Clubs\rangle\}\rangle, \\ & & \langle Ben, \{\langle 7, Diamonds\rangle, \langle 6, Spades\rangle\}\rangle, \\ & & \langle Col,\ \{\langle 7, Spades\rangle, \langle Q, Spades\rangle\}\rangle\} \end{array}$$

Give the expression that should appear in place of the dots $(\ldots)$.

(ii) [8 marks] Consider the following relation on $Suits$

$$\begin{aligned} \textit{WMT} &: Suits \leftrightarrow Suits \\ \textit{Gr(WMT)} &\triangleq \{\langle Spades, Hearts\rangle, \langle Hearts, Diamonds\rangle, \langle Diamonds, Clubs\rangle\} \end{aligned}$$

(WMT can be thought of as the 'worth more than' relation.)

(a) [3 marks] Carefully explain why WMT is *not* an *ordering*.

(b) [3 marks] Relation WMT' is just like WMT except that its graph has fewer tuples. Specifically:

$$Gr(WMT') = Gr(WMT) \setminus \{\langle Hearts, Diamonds\rangle\}$$

Is WMT' an *ordering*? **Justify** your answer.

(c) [2 marks] Enumerate the tuples in the graph of $RTC(WMT)$, the reflexive transitive closure of the WMT relation.

3 (20 marks)

(i) [3 marks] In this part of the question,

$$\begin{aligned} p &\triangleq \text{Ann is dealt the King of Hearts} \\ q &\triangleq \text{Ben is dealt the King of Hearts} \\ r &\triangleq \text{Col is dealt the Ace of Spades} \end{aligned}$$

Express the following statements as *well-formed formulae of propositional logic*.

(a) [1 mark] Ben is dealt the King of Hearts but Col is dealt the Ace of Spades.

(b) [2 marks] Ann is dealt the King of Hearts unless Ben is dealt the King of Hearts.

(ii) [4 marks] Draw the **full** *truth tables* of the following well-formed formulae of propositional logic. Hence, state whether each is *valid* or *invalid*, and *satisfiable* or *unsatisfiable*.

(a) [2 marks] $(p \vee q) \Rightarrow (p \Leftrightarrow q)$

(b) [2 marks] $(p \vee q) \vee \neg(p \wedge q)$

(iii) [5 marks]

(a) [$2\frac{1}{2}$ marks] Use a *truth table* to show that

$$\{(p \vee q) \Rightarrow r, p\} \models r$$

(b) [$2\frac{1}{2}$ marks] Use the rules of the *natural deduction* system given in the 'Examination Notes' to show that

$$\{(p \vee q) \Rightarrow r, p\} \vdash r$$

You **must** annotate each step of your derivation with the inference rule that the step uses.

(iv) [4 marks] In this part of the question, use the following 'key':

$$\begin{aligned} p(x) &: x \text{ is a player} \\ c(x) &: x \text{ is a card} \\ d(x, y) &: x \text{ is dealt } y \end{aligned}$$

(p and c are unary predicate symbols, and d is a binary predicate symbol.)

(a) [2 marks] Give a natural English paraphrase for the following formula of first-order predicate logic:

$$\exists x(c(x) \wedge \forall y(p(y) \Rightarrow \neg d(y, x)))$$

(b) [2 marks] Translate the following English statement into a well-formed formula of first-order predicate logic:

Every player is dealt at least one card.

(v) [4 marks] In this part of the question, take the universe of discourse $\mathcal{U}$ to be defined as follows:

$$\mathcal{U} \triangleq \{Q, K\}$$

Here is an interpretation $\mathcal{I}$ of the constant symbol a, the unary function symbol f and the binary predicate symbol p:

$\mathcal{I}(a)$	is Q
$\mathcal{I}(f)$	is the unary function on $\mathcal{U}$ whose maplets are $\{Q \mapsto K, K \mapsto K\}$
$\mathcal{I}(p)$	is the binary relation on $\mathcal{U}$ whose graph is $\{\langle Q, K\rangle, \langle K, Q\rangle\}$

Determine the truth value of the following well-formed formula. You **must** show your working.

$$\forall x \forall y (p(x, y) \Rightarrow p(a, y))$$

EXAMINATION NOTES

Set Theory

For any sets A, A_1, A_2 and A_3,...

Commutativity of $\cup$ and $\cap$

$$A_1 \cup A_2 = A_2 \cup A_1$$
$$A_1 \cap A_2 = A_2 \cap A_1$$

Associativity of $\cup$ and $\cap$

$$(A_1 \cup A_2) \cup A_3 = A_1 \cup (A_2 \cup A_3)$$
$$(A_1 \cap A_2) \cap A_3 = A_1 \cap (A_2 \cap A_3)$$

Distributivity of $\cup$ over $\cap$, and $\cap$ over $\cup$

$$A_1 \cup (A_2 \cap A_3) = (A_1 \cup A_2) \cap (A_1 \cup A_3)$$

$$A_1 \cap (A_2 \cup A_3) = (A_1 \cap A_2) \cup (A_1 \cap A_3)$$

De Morgan's laws

$$(A_1 \cup A_2)' = A_1' \cap A_2'$$
$$(A_1 \cap A_2)' = A_1' \cup A_2'$$

Turn over.

Idempotence of $\cup$ and $\cap$

$$A \cup A = A$$

$$A \cap A = A$$

Identities

$$\emptyset \cup A = A$$

$$U \cup A = U$$

$$\emptyset \cap A = \emptyset$$

$$U \cap A = A$$

Involution

$$(A')' = A$$

Complement laws

$$A \cup A' = U$$

$$A \cap A' = \emptyset$$

$$U' = \emptyset$$

$$\emptyset' = U$$

Logic

For any propositions W, W_1, W_2 and W_3,...

Commutativity of $\wedge$ and $\vee$

$$W_1 \wedge W_2 \equiv W_2 \wedge W_1$$
$$W_1 \vee W_2 \equiv W_2 \vee W_1$$

Associativity of $\wedge$ and $\vee$

$$(W_1 \wedge W_2) \wedge W_3 \equiv W_1 \wedge (W_2 \wedge W_3)$$
$$(W_1 \vee W_2) \vee W_3 \equiv W_1 \vee (W_2 \vee W_3)$$

Distributivity of $\wedge$ over $\vee$, and of $\vee$ over $\wedge$

$$W_1 \wedge (W_2 \vee W_3) \equiv (W_1 \wedge W_2) \vee (W_1 \wedge W_3)$$
$$W_1 \vee (W_2 \wedge W_3) \equiv (W_1 \vee W_2) \wedge (W_1 \vee W_3)$$

De Morgan's laws

$$\neg(W_1 \wedge W_2) \equiv \neg W_1 \vee \neg W_2$$
$$\neg(W_1 \vee W_2) \equiv \neg W_1 \wedge \neg W_2$$

Idempotence of $\wedge$ and $\vee$

$$W \wedge W \equiv W$$
$$W \vee W \equiv W$$

Turn over.

Identities

$$t \wedge W \equiv W$$

$$f \wedge W \equiv f$$

$$t \vee W \equiv t$$

$$f \vee W \equiv W$$

Involution

$$\neg\neg W \equiv W$$

Complement laws

$$W \wedge \neg W \equiv f$$

$$W \vee \neg W \equiv t$$

$$\neg t \equiv f$$

$$\neg f \equiv t$$

Definition of biconditional

$$W_1 \Leftrightarrow W_2 \equiv (W_1 \Rightarrow W_2) \wedge (W_2 \Rightarrow W_1)$$

Definition of conditional

$$W_1 \Rightarrow W_2 \equiv \neg W_1 \vee W_2$$

Contrapositive law

$$W_1 \Rightarrow W_2 \equiv \neg W_2 \Rightarrow \neg W_1$$

Definition of existential

$$\exists X W \equiv \neg \forall X \neg W$$

Definition of universal

$$\forall X W \equiv \neg \exists X \neg W$$

Some inference rules of a natural deduction system

$\wedge$-INTRODUCTION

$$\frac{W_1, W_2}{W_1 \wedge W_2}$$

$\wedge$-ELIMINATION-LEFT

$$\frac{W_1 \wedge W_2}{W_1}$$

$\wedge$-ELIMINATION-RIGHT

$$\frac{W_1 \wedge W_2}{W_2}$$

$\Rightarrow$-ELIMINATION

$$\frac{W_1, W_1 \Rightarrow W_2}{W_2}$$

$\vee$-INTRODUCTION-RIGHT

$$\frac{W_1}{W_1 \vee W_2}$$

$\vee$-INTRODUCTION-LEFT

$$\frac{W_2}{W_1 \vee W_2}$$

Turn over.

⇔-ELIMINATION-LEFT

$$\frac{W_1 \Leftrightarrow W_2, W_1}{W_2}$$

⇔-ELIMINATION-RIGHT

$$\frac{W_1 \Leftrightarrow W_2, W_2}{W_1}$$

¬-ELIMINATION

$$\frac{\neg\neg W}{W}$$

⇒-INTRODUCTION

$$\frac{\frac{W_1}{W_2}}{W_1 \Rightarrow W_2}$$

REPETITION

$$\frac{W}{W}$$