061311

THE UNIVERSITY of York

M.Sc., B.Sc. and B.A. Examinations 1995-96

DEPARTMENT OF COMPUTER SCIENCE

Part Ia, MScIPa

Mathematics for Information Processing

Time allowed: **One-and-a-half hours**

Silent non-programmable calculators may be used.
A set of 'Examination Notes' is attached.
Candidates should answer **all** questions from this paper.

In this paper the following set names are used throughout:

$\mathbb{N}$: the set of natural numbers (positive integers and zero)
$\mathbb{P}$: the set of positive integers (excludes zero)
$\mathbb{Z}$: the set of all integers (negative, zero and positive)

1 (10 marks)

Consider the following sets:

$$
\begin{aligned}
S_1 &\triangleq \{x \mid x \in \mathbb{N} \text{ and } x < 10 \text{ and } x \text{ is an odd number}\} \\
S_2 &\triangleq \{1, 2, 3, 5, 7, 11\}
\end{aligned}
$$

(i) [3 marks] Give *extensional* definitions of each of the following sets:

(a) [$\frac{1}{2}$ marks] $S_1 \cap S_2$

(b) [$\frac{1}{2}$ marks] $S_1 \cup S_2$

(c) [$\frac{1}{2}$ marks] $S_2 \setminus S_1$

(d) [$\frac{1}{2}$ marks] $S_1 \setminus S_2$

(e) [$\frac{1}{2}$ marks] $\wp(S_2 \setminus S_1)$

(f) [$\frac{1}{2}$ marks] $\wp(S_1 \setminus S_2)$

(ii) [2 marks] State whether the following are true or false:

(a) [$\frac{1}{2}$ marks] $S_2 \subseteq S_1$

(b) [$\frac{1}{2}$ marks] $\{3\} \in S_2$

(c) [$\frac{1}{2}$ marks] $\{7, 9\} \in \wp(S_1)$

(d) [$\frac{1}{2}$ marks] $\{7, 9\} \subseteq \wp(S_1)$

(iii) [5 marks] Prove the following algebraically using the laws given in the 'Examination Notes'. You **must** annotate each step of your proof with the algebraic law that is used in that step.

For any sets A_1 and A_2,

$$A_2 \cap ((A_2' \cap (A_1 \cap A_2)) \cup A_1) = A_1 \cap A_2$$

2 (20 marks)

(i) [10 marks] In this part of the question, P is a set of people and H is a relation from P to the positive integers $\mathbb{P}$ that tells us the height (in centimetres) of the people in the set P:

$$\begin{aligned} P &\triangleq \{\mathit{Ann}, \mathit{Ben}, \mathit{Col}, \mathit{Deb}, \mathit{Ern}\} \\ H &: P \leftrightarrow \mathbb{P} \\ Gr(H) &\triangleq \{\langle \mathit{Ann}, 137\rangle, \langle \mathit{Ben}, 152\rangle, \\ &\quad\ \langle \mathit{Col}, 137\rangle, \langle \mathit{Deb}, 140\rangle, \langle \mathit{Ern}, 137\rangle\} \end{aligned}$$

(a) [3 marks]

- Is H *total* or *partial*?
- Is it *surjective* or not?
- Is it *functional* or not?
- Is it *injective* or not?
- Is it *one-to-one, one-to-many, many-to-one* or *many-to-many*?
- Is it a *bijection*?

(b) [7 marks] Consider the following two relations on P:

$$\begin{aligned} \mathit{SHA} &: P \leftrightarrow P \\ Gr(\mathit{SHA}) &\triangleq \{\langle a, b\rangle \mid a \text{ is the same height as } b\} \\ \mathit{TT} &: P \leftrightarrow P \\ Gr(\mathit{TT}) &\triangleq \{\langle a, b\rangle \mid a \text{ is taller than } b\} \end{aligned}$$

- Give *extensional* definitions of the graphs of these two relations. (1 mark)

- State whether *SHA* and *TT* are *equivalence relations* or *orderings*. **Justify** your answers. (4 marks)
- If either is an equivalence relation, give the *partition* it defines on P. If either is an ordering, state whether the ordering is *total* or *partial*, and whether it is *strict* or not. (2 marks)

(ii) [5 marks] The set T is a set of towns. We define a binary relation R on T as follows:

$$\begin{aligned} R &: T \leftrightarrow T \\ Gr(R) &\triangleq \{\langle a, b\rangle \mid \text{the Buggy Bus Company operates} \\ &\qquad\qquad \text{a direct bus route from town } a \text{ to town } b\} \end{aligned}$$

This means, for example, that if *Manchester* and *Leeds* are members of T and if

$$\langle \textit{Manchester, Leeds} \rangle \in Gr(R)$$

then a Buggy Bus can take us from Manchester to Leeds (without changing bus).

Assuming that $\textit{York} \in T$, write expressions that, when evaluated, would compute

(a) [1 mark] the set of towns that can be reached on a Buggy Bus route *from* York (without changing bus);

(b) [1 mark] the set of towns from which one can travel on a Buggy Bus route *to* York (without changing bus);

(c) [1 mark] the set of towns which it is *not* possible to leave on a Buggy Bus;

(d) [1 mark] the relation on T whose graph maps a town to another town that can be reached by a Buggy Bus journey that includes *one* change of bus;

(e) [1 mark] the relation on T whose graph maps a town to another town that can be reached by a Buggy Bus journey (of any number of changes).

(iii) [5 marks] Let the functions f_1 and f_2 be defined as follows:

$$\begin{aligned} f_1 &: \mathbb{N} \to \mathbb{N} \\ f_1(x) &\triangleq (2 * x) + 1 \\ f_2 &: \mathbb{N} \to \mathbb{Z} \\ f_2(x) &\triangleq x - 2 \end{aligned}$$

(a) [$1\frac{1}{2}$ marks] State whether f_1 and f_2 are *injective* or not. If you think either is *not*, give an example to show this.

(b) [$1\frac{1}{2}$ marks] State whether f_1 and f_2 are *surjective* or not. If you think either is *not*, give an example to show this.

(c) [1 mark] The function f_3 is equal to the *composition* $f_2 \circ f_1$. Give a definition of f_3.

(d) [1 mark] The function f_4 is equal to the *inverse* of f_2, f_2^{-1}. Give a definition of f_4.

3 (20 marks)

(i) [2 marks] In this part of the question,

$$\begin{aligned} p &\triangleq \text{The weather is pleasant} \\ q &\triangleq \text{It is foggy} \\ r &\triangleq \text{It is raining} \end{aligned}$$

Express the following statements as *well-formed formulae of propositional logic.*

(a) [1 mark] The weather is unpleasant precisely when it is foggy and/or raining.

(b) [1 mark] Unless the weather is pleasant, it is foggy or raining.

(ii) [6 marks] Draw the **full** *truth tables* of the following well-formed formulae of propositional logic. Hence, state whether each is *valid* or *invalid*, and *satisfiable* or *unsatisfiable*.

(a) [2 marks] $(p \wedge q) \Rightarrow (p \Leftrightarrow q)$

(b) [2 marks] $((\neg p \Rightarrow q) \wedge p) \Rightarrow \neg q$

(c) [2 marks] $(p \wedge q) \wedge \neg(p \vee q)$

(iii) [4 marks]

(a) [2 marks] Use a *truth table* to show that

$$\{q \Rightarrow \neg p, q \wedge r\} \models \neg p$$

(b) [2 marks] Use the rules of the *natural deduction* system given in the 'Examination Notes' to show that

$$\{q \Rightarrow \neg p, q \wedge r\} \vdash \neg p$$

You **must** annotate each step of your derivation with the inference rule that the step uses.

(iv) [4 marks] In this part of the question, use the following 'key':

$$\begin{aligned} s(x) &: x \text{ is a student} \\ e(x) &: x \text{ is an exam} \\ p(x, y) &: x \text{ will pass } y \end{aligned}$$

(s and e are unary predicate symbols, and p is a binary predicate symbol.)

(a) [2 marks] Give a natural English paraphrase for the following formula of first-order predicate logic:

$$\exists x \exists y((e(x) \wedge s(y) \wedge p(y, x)) \Leftrightarrow \forall z(s(z) \Rightarrow p(z, x)))$$

(b) [2 marks] Translate the following English statements into well-formed formulae of first-order predicate logic:

- Every student will pass at least one exam.
- At least one student will pass all exams.

(v) [4 marks] In this part of the question, take the universe of discourse $\mathcal{U}$ to be defined as follows:

$$\mathcal{U} \triangleq \{1, 2\}$$

Here is an interpretation $\mathcal{I}$ of the constant symbol a, the unary function symbol f and the binary predicate symbol p:

$\mathcal{I}(a)$	is 1
$\mathcal{I}(f)$	is the unary function on $\mathcal{U}$ whose maplets are $\{1 \mapsto 2, 2 \mapsto 1\}$
$\mathcal{I}(p)$	is the binary relation on $\mathcal{U}$ whose graph is $\{\langle 1, 1\rangle, \langle 2, 2\rangle\}$

Determine the truth values of the following well-formed formulae. You **must** show your working.

(a) [$1\frac{1}{2}$ marks] $\forall x \; p(x, f(f(x)))$

(b) [$2\frac{1}{2}$ marks] $\forall x \forall y (p(x, y) \Rightarrow p(a, y))$

EXAMINATION NOTES

Set Theory

For any sets A, A_1, A_2 and A_3,...

Commutativity of $\cup$ and $\cap$

$$A_1 \cup A_2 = A_2 \cup A_1$$
$$A_1 \cap A_2 = A_2 \cap A_1$$

Associativity of $\cup$ and $\cap$

$$(A_1 \cup A_2) \cup A_3 = A_1 \cup (A_2 \cup A_3)$$
$$(A_1 \cap A_2) \cap A_3 = A_1 \cap (A_2 \cap A_3)$$

Distributivity of $\cup$ over $\cap$, and $\cap$ over $\cup$

$$A_1 \cup (A_2 \cap A_3) = (A_1 \cup A_2) \cap (A_1 \cup A_3)$$

$$A_1 \cap (A_2 \cup A_3) = (A_1 \cap A_2) \cup (A_1 \cap A_3)$$

De Morgan's laws

$$(A_1 \cup A_2)' = A_1' \cap A_2'$$
$$(A_1 \cap A_2)' = A_1' \cup A_2'$$

Idempotence of $\cup$ and $\cap$

$$A \cup A = A$$

$$A \cap A = A$$

Identities

$$\emptyset \cup A = A$$

$$U \cup A = U$$

$$\emptyset \cap A = \emptyset$$

$$U \cap A = A$$

Involution

$$(A')' = A$$

Complement laws

$$A \cup A' = U$$

$$A \cap A' = \emptyset$$

$$U' = \emptyset$$

$$\emptyset' = U$$

Turn over.

Logic

For any propositions W, W_1, W_2 and W_3,…

Commutativity of $\wedge$ and $\vee$

$$W_1 \wedge W_2 \equiv W_2 \wedge W_1$$
$$W_1 \vee W_2 \equiv W_2 \vee W_1$$

Associativity of $\wedge$ and $\vee$

$$(W_1 \wedge W_2) \wedge W_3 \equiv W_1 \wedge (W_2 \wedge W_3)$$
$$(W_1 \vee W_2) \vee W_3 \equiv W_1 \vee (W_2 \vee W_3)$$

Distributivity of $\wedge$ over $\vee$, and of $\vee$ over $\wedge$

$$W_1 \wedge (W_2 \vee W_3) \equiv (W_1 \wedge W_2) \vee (W_1 \wedge W_3)$$
$$W_1 \vee (W_2 \wedge W_3) \equiv (W_1 \vee W_2) \wedge (W_1 \vee W_3)$$

De Morgan's laws

$$\neg(W_1 \wedge W_2) \equiv \neg W_1 \vee \neg W_2$$
$$\neg(W_1 \vee W_2) \equiv \neg W_1 \wedge \neg W_2$$

Idempotence of $\wedge$ and $\vee$

$$W \wedge W \equiv W$$
$$W \vee W \equiv W$$

Identities

$$t \wedge W \equiv W$$

$$f \wedge W \equiv f$$

$$t \vee W \equiv t$$

$$f \vee W \equiv W$$

Involution

$$\neg\neg W \equiv W$$

Complement laws

$$W \wedge \neg W \equiv f$$

$$W \vee \neg W \equiv t$$

$$\neg t \equiv f$$

$$\neg f \equiv t$$

Definition of biconditional

$$W_1 \Leftrightarrow W_2 \equiv (W_1 \Rightarrow W_2) \wedge (W_2 \Rightarrow W_1)$$

Definition of conditional

$$W_1 \Rightarrow W_2 \equiv \neg W_1 \vee W_2$$

Turn over.

Contrapositive law

$$W_1 \Rightarrow W_2 \equiv \neg W_2 \Rightarrow \neg W_1$$

Definition of existential

$$\exists X W \equiv \neg \forall X \neg W$$

Definition of universal

$$\forall X W \equiv \neg \exists X \neg W$$

Some inference rules of a natural deduction system

$\wedge$-INTRODUCTION

$$\frac{W_1, W_2}{W_1 \wedge W_2}$$

$\wedge$-ELIMINATION-LEFT

$$\frac{W_1 \wedge W_2}{W_1}$$

$\wedge$-ELIMINATION-RIGHT

$$\frac{W_1 \wedge W_2}{W_2}$$

$\Rightarrow$-ELIMINATION

$$\frac{W_1, W_1 \Rightarrow W_2}{W_2}$$

$\vee$-INTRODUCTION-RIGHT

$$\frac{W_1}{W_1 \vee W_2}$$

$\vee$-INTRODUCTION-LEFT

$$\frac{W_2}{W_1 \vee W_2}$$

$\Leftrightarrow$-ELIMINATION-LEFT

$$\frac{W_1 \Leftrightarrow W_2, W_1}{W_2}$$

$\Leftrightarrow$-ELIMINATION-RIGHT

$$\frac{W_1 \Leftrightarrow W_2, W_2}{W_1}$$

$\neg$-ELIMINATION

$$\frac{\neg\neg W}{W}$$

$\Rightarrow$-INTRODUCTION

$$\frac{\frac{W_1}{W_2}}{W_1 \Rightarrow W_2}$$

REPETITION

$$\frac{W}{W}$$