Clausal form logic

- A canonical/ normal/ standard form for wffs.
- As expressive as first-order predicate calculus.
- But FOPC and corresponding CFL are not logically equivalent
  - preserves (un)satisfiability
  - our proof theory is based on showing inconsistency (unsatisfiability)
- For CFL, there is a sound and complete proof theory
  - only one inference rule (resolution)
  - no logical axiom schemata

Conversion from FOPC to CFL

1. Relabel clashing variables.
2. Eliminate $\rightarrow$ and $\leftrightarrow$.
3. Move $\neg$ inwards.
4. Skolemise existentially quantified variables.
5. Drop universal quantifiers.
6. Distribute disjunction over conjunction.
7. Rewrite as clauses.
8. Standardise variables apart.

— *in that order.*

1. Relabel clashing variables

$$\forall X \forall Y \neg(p(X,Y) \rightarrow \forall Y q(Y,Y))$$

becomes

$$\forall X \forall Y \neg(p(X,Y) \rightarrow \forall Z q(Z,Z))$$

(Better still, avoid clashing variables in the first place.)

2. Eliminate $\rightarrow$ and $\leftrightarrow$

$$\begin{array}{lcl}(\alpha \rightarrow \beta) & \equiv & (\neg\alpha \vee \beta) \\ (\alpha \leftrightarrow \beta) & \equiv & (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)\end{array}$$

$$\forall X \forall Y \neg(p(X,Y) \rightarrow \forall Z q(Z,Z))$$

becomes

$$\forall X \forall Y \neg(\neg p(X,Y) \vee \forall Z q(Z,Z))$$

3. Move ¬ 'inwards' until the only expressions being negated are atoms.

$$\begin{array}{rcl} \neg\forall X\alpha & \equiv & \exists X\neg\alpha \\ \neg\exists X\alpha & \equiv & \forall X\neg\alpha \\ \neg(\alpha\vee\beta) & \equiv & (\neg\alpha\wedge\neg\beta) \\ \neg(\alpha\wedge\beta) & \equiv & (\neg\alpha\vee\neg\beta) \\ \neg\neg\alpha & \equiv & \alpha \end{array}$$

$$\forall X\forall Y\neg(\neg p(X,Y)\vee\forall Z q(Z,Z))$$

becomes

$$\forall X\forall Y(\neg\neg p(X,Y)\wedge\neg\forall Z q(Z,Z))$$

which then becomes

$$\forall X\forall Y(p(X,Y)\wedge\exists Z\neg q(Z,Z))$$

4. Skolemise existentially quantified variables

Schematically,

a. $\exists Y\forall X_1,\ldots,\forall X_n p(X_1,\ldots,X_n,Y)$
becomes
$\forall X_1,\ldots,\forall X_n p(X_1,\ldots,X_n,c)$

b. $\forall X_1,\ldots,\forall X_n\exists Y p(X_1,\ldots,X_n,Y)$
becomes
$\forall X_1,\ldots,\forall X_n p(X_1,\ldots,X_n,f(X_1,\ldots,X_n))$

where c and f are new constants/function symbols

$$\forall X\forall Y(p(X,Y)\wedge\exists Z\neg q(Z,Z))$$

becomes

$$\forall X\forall Y(p(X,Y)\wedge\neg q(f(X,Y),f(X,Y)))$$

Another example

$$\exists W(p(W)\wedge\forall X\exists Y\exists Z q(W,X,Y,Z))$$

becomes

$$p(c)\wedge\forall X q(c,X,g(X),h(X))$$

5. Drop universal quantifiers

$$\forall X\forall Y(p(X,Y)\wedge\neg q(f(X,Y),f(X,Y)))$$

becomes

$$p(X,Y)\wedge\neg q(f(X,Y),f(X,Y))$$

6. Distribute disjunction over conjunction

—will end up in *conjunctive normal form*: a conjunction of disjunctions, i.e. something of the form

$$(P_1\vee P_2\vee\ldots\vee P_l)\wedge(Q_1\vee Q_2\vee\ldots\vee Q_m)\wedge\ldots$$
$$\ldots\wedge(R_1\vee R_2\vee\ldots\vee R_n)$$

where each P_i, Q_i and R_i is an atom or a negated atom.

$$p(X,Y)\wedge\neg q(f(X,Y),f(X,Y))$$

—no change

CNF continued

$$\begin{array}{lcl}(\alpha \vee (\beta \wedge \gamma)) & \equiv & ((\alpha \vee \beta) \wedge (\alpha \vee \gamma)) \\ ((\beta \wedge \gamma) \vee \alpha) & \equiv & ((\beta \vee \alpha) \wedge (\gamma \vee \alpha))\end{array}$$

Also, associativity can be useful:

$$\begin{array}{lclcl}(\alpha \vee (\beta \vee \gamma)) & \equiv & ((\alpha \vee \beta) \vee \gamma) & \equiv & \alpha \vee \beta \vee \gamma \\ (\alpha \wedge (\beta \wedge \gamma)) & \equiv & ((\alpha \wedge \beta) \wedge \gamma) & \equiv & \alpha \wedge \beta \wedge \gamma\end{array}$$

$$p \vee (q \wedge r \wedge s \wedge (t \vee u))$$

becomes

$$(p \vee q) \wedge (p \vee r) \wedge (p \vee s) \wedge (p \vee t \vee u)$$

$$p \wedge (q \vee r \vee s \vee (t \wedge u))$$

becomes

$$p \wedge (q \vee r \vee s \vee t) \wedge (q \vee r \vee s \vee u)$$

7. Rewrite as clauses

Each conjunct is a clause.

$$(P_1 \vee \ldots P_l) \wedge (Q_1 \vee \ldots \vee Q_m) \wedge (R_1 \vee \ldots \vee R_n)$$

becomes

$$P_1 \vee \ldots \vee P_l$$

$$Q_1 \vee \ldots \vee Q_m$$

$$R_1 \vee \ldots \vee R_n$$

$$p(X, Y) \wedge \neg q(f(X, Y), f(X, Y))$$

becomes

$$p(X, Y)$$

$$\neg q(f(X, Y), f(X, Y))$$

8. Standardise *variables* in different clauses apart.

$$p(X, Y)$$

$$\neg q(f(X, Y), f(X, Y))$$

become

$$p(X_1, Y_1)$$

$$\neg q(f(X_2, Y_2), f(X_2, Y_2))$$

English	Every student uses some computer, but at least one (specific) computer is used by every student.	
Key	$s(X)$	X is a student
	$c(X)$	X is a computer
	$u(X, Y)$	X uses Y
FOPC		

1. Relabel clashing variables

2. Eliminate $\rightarrow$ and $\leftrightarrow$

3. Move $\neg$ inwards

4. Skolemise existentially quantified variables

5. Drop universal quantifiers

6. Distribute disjunction over conjunction (to give a conjunction of disjunctions)

7. Rewrite as clauses

8. Standardise variables apart

CFL

- A **literal** is an atom or the negation of an atom.
- A **clause** is therefore a disjunction of literals.
- Clauses can be thought of as *sets* of literals
  - so order of literals is irrelevant
  - and can remove duplicates
- The **empty clause** is written $\Box$.
- A **ground clause**/ **literal**/ **atom**/ **term**/ **wff** is one without variables.

Resolution

- Our only inference rule.
- Example
  1. $p_1 \vee p_2$
  2. $\neg p_1 \vee p_3$
  3. $p_2 \vee p_3$ (from resolution of 1 and 2)

  - (1) and (2) are the **parents**
  - p_1 and $\neg p_1$ are the **selected literals**
  - p_1 and $\neg p_1$ are a **complementary pair**
  - (3) is the **resolvent**

Resolution continued

- Depict in tree form:

- Soundness

  i.e. $\{p_1 \vee p_2, \neg p_1 \vee p_3\} \models p_2 \vee p_3$

p_1	p_2	p_3	$p_1 \vee p_2$	$\neg p_1 \vee p_3$	$p_2 \vee p_3$	
T	T	T	T	T	T	*
T	T	F	T	F	T	
T	F	T	T	T	T	*
T	F	F	T	F	F	
F	T	T	T	T	T	*
F	T	F	T	T	T	*
F	F	T	F	T	T	
F	F	F	F	T	F	

Some examples

- Clauses are sets, so duplicates can be removed from resolvents:

  1. $p \vee q$
  2. $\neg p \vee q$
  3.

- Modus ponens is a special case of resolution:

  1. p 1.
  2. $p \to q$ 2.
  3. q 3.

More examples

- Modus tollens is a special case of resolution:

  1. $\neg q$ 1.
  2. $p \to q$ 2.
  3. $\neg p$ 3.

- Resolving two unit clauses gives the empty clause:

  1. p
  2. $\neg p$
  3.

Examples requiring care

- Example 1
  1. $p \vee q$
  2. $\neg p \vee \neg q$
  3. (from resolution of 1 and 2)
  4. (from resolution of 1 and 2)

- Example 2
  1. $p \vee \neg q$
  2. $\neg p \vee q$
  3. (from resolution of 1 and 2)
  4. (from resolution of 1 and 2)

The empty clause is *not* a resolvent in either example. In each example, there are *two* resolvents, both simplifying to true.

Exercises

- Exercise 1
  1. $p \vee \neg q \vee r$
  2. $p \vee \neg r \vee q \vee s$
  3.
  4.

- Exercise 2
  1. $p \vee p$
  2. $\neg p \vee \neg p$
  3.

Bindings and substitutions

- A **binding**

$$V \mapsto e$$

  - variable V is bound to expression e
  - e may be a constant, another variable, or a function symbol with its arguments
    (but e cannot contain V, e.g. $X \mapsto f(X)$ is illegal)

- A **substitution**
  - a set of bindings
  - e.g. $\{X \mapsto a, Y \mapsto Z, Z \mapsto f(a,b)\}$

Applying a substitution

Replace variables in the expression by what they are bound to.

E.g. let e be $p(X, f(Y), b)$

θ	$e\theta$
$\{X \mapsto Z, Y \mapsto W\}$	
$\{Y \mapsto c\}$	
$\{X \mapsto g(Z), Y \mapsto d\}$	
$\{X \mapsto c, Y \mapsto d\}$	

Composing two substitutions

Apply the 2nd set to each expression in the 1st set, then add bindings from the 2nd set whose variables do not appear among the variables of the 1st set.

θ $\{Z \mapsto f(X,Y)\}$

τ $\{X \mapsto b, Y \mapsto c, W \mapsto d, Z \mapsto e\}$

$\theta\tau$

θ $\{X_1 \mapsto f(Y_1), X_2 \mapsto Y_2, X_3 \mapsto g(Y_1, Y_2)\}$

τ $\{Y_1 \mapsto a, Y_2 \mapsto Y_3\}$

$\theta\tau$

Unification

- Two expressions, e_1 and e_2, **unify** if there exists a substitution θ such that
$$e_1\theta = e_2\theta$$
then we say that θ is a unifier of e_1 and e_2.

- E.g.

  $p(a, X, f(Y), b)$

  $p(Z, Z, f(c), b)$

Most general unifier (mgu)

- One unifier

  $$p(X, f(Y), b)\,\{X \mapsto c, Y \mapsto b\} \quad =$$
  $$p(X, f(b), b)\,\{X \mapsto c, Y \mapsto b\} \quad =$$

- Another unifier

  $$p(X, f(Y), b)\,\{Y \mapsto b\} \quad =$$
  $$p(X, f(b), b)\,\{Y \mapsto b\} \quad =$$

- $\{Y \mapsto b\}$ is more general than $\{X \mapsto c, Y \mapsto b\}$

Resolution with variables

- Clauses can be resolved if there is some pair in the parents that is complementary when some mgu is applied to them.

- The mgu is also applied to the resolvent.

- E.g.
  1. $p(X) \vee q(X, Y)$
  2. $\neg p(a) \vee r(b, Z)$
  3.

Exercises

- Exercise 1
  1. $p(X) \vee q(X, Y)$
  2. $\neg p(f(Z)) \vee \neg r(f(a), b) \vee s(Z)$
  3.

- Exercise 2
  1. $p(X, a) \vee q(X, Y)$
  2. $\neg p(f(Z), Z) \vee r(Z)$
  3.

Resolution

- In general, let α, β and γ be clauses. Then, γ is a resolvent of α and β iff
  - there is a non-empty set of *positive* literals
    $$A \subseteq \alpha$$
    and a non-empty set of *negative* literals
    $$B \subseteq \beta$$
    such that A and the complement of the literals in B unify with some mgu θ;
  - $\gamma = ((\alpha - A) \cup (\beta - B))\theta$
- E.g.
  1. $p(U) \vee p(V)$
  2. $\neg p(X) \vee \neg p(Y)$
  3.

Resolution deduction

(1) English: All elephants are grey

FOPC:

(2) English: Clyde is an elephant

FOPC:

To show

(3) English: Clyde is grey

FOPC:

(1)

(2)

(3)

Resolution deduction is incomplete

- A case where $\Phi \models \alpha$ but $\Phi \not\vdash \alpha$
- $\{$*elephant(clyde),* $\neg$*elephant(clyde)*$\} \models$ *mouse(martie)*
- Resolution deduction does not derive *mouse(martie)*

  1.

  2.

  3.
- For completeness, we would have to add some *logical axiom schemata* to our proof theory.
- Fortunately, there is an alternative.

Resolution refutation

- Instead of using resolution to show
  $$\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \vdash \beta$$
  we use resolution to show that
  $$\alpha_1 \wedge \alpha_2 \wedge \ldots \alpha_n \wedge \neg\beta$$
  is **inconsistent.**
  - proof by contradiction/refutation proof
  - i.e. try to derive $\Box$ from
    $$\{\alpha_1, \alpha_2, \ldots, \alpha_n, \neg\beta\}$$
    If you succeed then
    $$\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \models \beta$$
    Otherwise
    $$\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \not\models \beta$$
- Resolution refutation on clausal form is complete.

Resolution refutation

1. Convert all the *premisses* to clausal form.

2. Negate the goal <u>*and then*</u> convert to clausal form.

3. Repeat until either a contradiction is found, no progress can be made or a predetermined amount of effort has been expended

   - Select two clauses (the parents)

   - Resolve them together

   - If one of the resolvents is the empty clause, then a contradiction has been found
     If not, standardise the variables apart in these new clauses and then add them to the set of clauses available to the procedure.

(Show the proof in the form of a **refutation tree**.)

1. Every elephant is grey.
2. Clyde is an elephant.
3. Is Clyde grey?

1. If one is in Paris, then one is not in Moscow.
2. Flopsy is in Paris.
3. Is Flopsy in Moscow?

Exercise

1. Everyone who saves money earns interest.
2. Show if there is no interest earned, then nobody saves money.

Use $s(X)$ for X saves money and $e(X)$ for X earns interest.

1. Flopsy is Clyde's friend.
2. Who is Clyde's friend?

1. For all X, Y and Z, if X is the father of Y and Z is the father of X, then Z is the grandfather of Y.
2. Everyone has a father.
3. For some X, who is the grandfather of X?