Lecture 5:
Formal Deduction

Aims:

- To discuss what we mean by a deduction system;
- To look at the inference rules of a natural deduction system for propositional logic.

5.1 Deduction Systems

- To show $\{W_1, W_2, \ldots, W_n\} \models W$, we must check all interpretations of the wffs, i.e. we must draw a truth table for $\{W_1, W_2, \ldots, W_n\} \models W$

  Checking all interpretations is manageable for wffs that have 2 or 3 proposition symbols, but becomes increasingly difficult when there are more.

- This motivates the development of apparatus that enables reasoning to take place at the purely syntactic level, before interpretations are considered.

  The apparatus is called a *(formal) deduction system*, a *proof theory*, an *inference system* or a *logical calculus*.

  Such systems can be broadly classified into deduction systems and refutation systems. We look at the former here.

- A *deduction system* enables us to derive conclusions from a set of wffs by *syntactic operations* alone. We manipulate the set of wffs without reference to their semantics. If the manipulation rules are 'right' (in a sense to be explored later), the new wffs we generate will in fact be logical consequences of the original set of wffs. The process is 'mechanical' and thus amenable to automation.

- A deduction system comprises a finite set of *inference rules* and a set of wffs called *logical axioms*. The latter set is often infinite, but this is not a problem as one can use templates for these wffs, these templates being called *logical axiom schemata*. In fact, the deduction system we present below has no logical axioms, so we'll not discuss these any further.

- An *inference rule* comprises a set of patterns called the *conditions*, and another pattern called the *conclusion*.

  E.g.

  $\Rightarrow$-ELIMINATION

  $$\frac{W_1, W_1 \Rightarrow W_2}{W_2}$$

  Above the line are the conditions, below is the conclusion. If you have wffs that match the conditions, then you can, in a single step, infer a wff that matches the conclusion.

Suppose we have wffs p and $p \Rightarrow q$, then using the inference rule we can infer q. We set this out as follows:

1.	p	premiss
2.	$p \Rightarrow q$	premiss
3.	q	from 1,2 by $\Rightarrow$-ELIMINATION

Now suppose we have the wff $q \Rightarrow r$ as well as p and $p \Rightarrow q$. Can we infer r?

(We 'ought' to be able to if this deduction system is any good, because r is certainly a *logical consequence* of the premisses, i.e. $\{p, p \Rightarrow q, q \Rightarrow r\} \models r$, as you can confirm for yourself with a truth table.)

But, using the inference rule, we cannot infer r in a single step. To remedy this, we could introduce another inference rule, e.g.

$$\frac{W_1, W_1 \Rightarrow W_2, W_2 \Rightarrow W_3}{W_3}$$

But, while some additional inference rules would be useful, we cannot keep introducing them in this way: we would need an infinite number of them.

- We allow *sequences* of inferences that might make use of the same inference rule repeatedly on the newly generated wffs:

1.	p	premiss
2.	$p \Rightarrow q$	premiss
3.	$q \Rightarrow r$	premiss
4.	q	from 1,2 by $\Rightarrow$-ELIMINATION
5.	r	from 3,4 by $\Rightarrow$-ELIMINATION

  In applying an inference rule, we allow ourselves to match its conditions with premisses *or* with wffs inferred earlier. (In fact, in a system that allows logical axioms you can use these too, and, as we will see, 'assumptions' can also be made and matched with the conditions of inference rules.)

- Such a sequence is called a *deduction* or a *derivation*.

- If A is the set of premisses and W is the *final* conclusion, then we write

$$A \vdash W$$

  We say that W is *deducible* or *derivable* from A.

  $\vdash$ is called the 'syntactic turnstile' symbol, and is a metasymbol.

- There are many choices of deduction system. They offer different sets of logical axioms (including none) and different sets of inference rules.

  We look at one example of what is called a *natural deduction* system.

5.2 A Natural Deduction System

- The idea is to provide a rule of inference to 'introduce' and 'eliminate' each of the five connectives.

- The $\wedge$-INTRODUCTION rule says that if wffs W_1 and W_2 are among the premisses or have been derived earlier, then you can infer wff $W_1 \wedge W_2$:

  $\wedge$-INTRODUCTION

$$\frac{W_1, W_2}{W_1 \wedge W_2}$$

- The $\wedge$-ELIMINATION rules say that if you have the wff $W_1 \wedge W_2$ among the premisses or you have derived them earlier, you may infer W_1 on its own and/or W_2 on its own:

  $\wedge$-ELIMINATION-LEFT
  $$\frac{W_1 \wedge W_2}{W_1}$$

  $\wedge$-ELIMINATION-RIGHT
  $$\frac{W_1 \wedge W_2}{W_2}$$

  There are lots more rules yet, but to make matters clearer, we immediately show an example that uses the rules just given.

- E.g. show that $\{p \wedge q\} \vdash q \wedge p$

1.	$p \wedge q$	premiss
2.	p	1, $\wedge$-E-L
3.	q	1, $\wedge$-E-R
4.	$q \wedge p$	2,3, $\wedge$-I

  I've abbreviated the explanations in the rightmost column.

- E.g. show that $\{p \wedge (q \wedge r)\} \vdash (p \wedge q) \wedge r$

  We'll do this example in the lecture.

  1.
  2.
  3.
  4.
  5.
  6.
  7.

- Finding a derivation is a process of trial and error: often, it will be possible to apply several rules, and you will have to make a choice. If that choice doesn't 'work out', you'll have to go back and try something else. Do not be discouraged by this process. It happens to *all* of us!

- The next rule, $\Rightarrow$-ELIMINATION, is the rule we saw earlier. Some books call it *modus ponens*.

  $\Rightarrow$-ELIMINATION
  $$\frac{W_1, W_1 \Rightarrow W_2}{W_2}$$

- Now we look at $\vee$-INTRODUCTION-RIGHT and $\vee$-INTRODUCTION-LEFT.

  If W_1 is among the premisses or has been derived earlier, then you can infer $W_1 \vee W_2$. This may seem unnatural. As far as these course notes are concerned, all I will say by way of justification of this rule is that inference rules are syntactic operations, and we can have them do whatever we want. Of course, if derivation is to have any 'tie in' with logical consequence, then we will in fact need the 'right' rules. This is a topic we return to later. I will attempt to explain informally why the following rules are 'right' in the lecture.

  $\vee$-INTRODUCTION-RIGHT
  $$\frac{W_1}{W_1 \vee W_2}$$

  $\vee$-INTRODUCTION-LEFT
  $$\frac{W_2}{W_1 \vee W_2}$$

- And here are three more rules:

  $\Leftrightarrow$-ELIMINATION-LEFT
  $$\frac{W_1 \Leftrightarrow W_2, W_1}{W_2}$$

  $\Leftrightarrow$-ELIMINATION-RIGHT
  $$\frac{W_1 \Leftrightarrow W_2, W_2}{W_1}$$

  $\neg$-ELIMINATION
  $$\frac{\neg\neg W}{W}$$

- Here are some examples.

  Show that $\{p \wedge q\} \vdash p \vee q$

1.	$p \wedge q$	premiss
2.	p	1, $\wedge$-E-L
3.	$p \vee q$	2, $\vee$-I-R

  Show that $\{p, q, (p \wedge q) \Rightarrow r\} \vdash r$

1.	p	premiss
2.	q	premiss
3.	$(p \wedge q) \Rightarrow r$	premiss
4.	$p \wedge q$	1,2, $\wedge$-I
5.	r	3,4, $\Rightarrow$-E

- More complicated inference rules make use of *subderivations*. Roughly, the idea is that, within some derivation, we might have a subderivation of some wff, this subderivation making use of an *assumption* of some wffs. In other words, within the subderivation, new wffs are derived from the premisses, wffs that have been derived earlier and the assumption(s). When we 'come out of' the subderivation, we *discharge* the assumption: it cannot be used in any further steps. But, the more complicated inference rules that we are about to see state that when a subderivation of some wff has been found, when we 'come out of' the subderivation, we can infer some other wff (often one that mentions the assumption).

  It is important not to confuse assumptions with premisses: once discharged, assumptions and other wffs derived *within* the subderivation cannot be used in later steps of the derivation.

- The condition of the $\Rightarrow$-INTRODUCTION rule requires a subderivation of W_2 from assumption W_1. If such a subderivation can be found, then the rule allows us to discharge the assumption and infer the wff $W_1 \Rightarrow W_2$.

  $\Rightarrow$-INTRODUCTION
  $$\frac{\frac{W_1}{W_2}}{W_1 \Rightarrow W_2}$$

- Example: we show that $\{p \Rightarrow q, q \Rightarrow r\} \vdash p \Rightarrow r$

1.	$p \Rightarrow q$		premiss
2.	$q \Rightarrow r$		premiss
3.	Subderivation:		
	3.1	p	assumption
	3.2	q	1, 3.1, $\Rightarrow$-E
	3.3	r	2, 3.2, $\Rightarrow$-E
4.	$p \Rightarrow r$		3, $\Rightarrow$-I

- We'll do this one in the lecture.

  Show that $\{p \Rightarrow q, p \Rightarrow r\} \vdash p \Rightarrow (q \wedge r)$

  1.
  2.
  3.
     3.1
     3.2
     3.3
     3.4
  4.

- Here are more rules.

  $\vee$-ELIMINATION

$$\frac{W_1 \vee W_2, \frac{W_1}{W_3}, \frac{W_2}{W_3}}{W_3}$$

  $\Leftrightarrow$-INTRODUCTION

$$\frac{\frac{W_1}{W_2}, \frac{W_2}{W_1}}{W_1 \Leftrightarrow W_2}$$

  $\neg$-INTRODUCTION

$$\frac{\frac{W_1}{W_2 \wedge \neg W_2}}{\neg W_1}$$

- One final rule, which is not really essential, but can make derivations easier to read is:

  REPETITION

$$\frac{W}{W}$$

  This simply allows us to write a wff again later in a derivation.