Lecture 4:
Propositional Logic Metatheoretic Definitions

Aims:

- To discuss what we mean by valid, invalid, satisfiable and unsatisfiable;
- To discuss what we mean by logical consequence;
- To discuss what we mean by logical equivalence; and
- To look at laws of propositional logic.

4.1 Valid, Invalid, Satisfiable and Unsatisfiable

- The truth-tables we saw in the previous lecture show that some wffs evaluate to true no matter how their atomic wffs are interpreted (every interpretation satisfies them), some evaluate to false for every interpretation of their atomic wffs (no interpretation satisfies them), and others are true for some interpretations and false for others (some interpretations satisfy them, others don't.) There is some terminology for all of this. (There would be, wouldn't there?)

always true	sometimes true/sometimes false	always false
tautology	contingency	contradiction
logically true	contingency	logically false
valid	invalid	
unfalsifiable	falsifiable	
satisfiable		unsatisfiable

Some books also use 'consistent' for 'satisfiable' and 'inconsistent' for 'unsatisfiable' but this is to be avoided.

We will use just the following: valid/invalid and satisfiable/unsatisfiable. I ignore all the others to avoid terminology overload:

- A wff is *valid* if all interpretations satisfy it. Otherwise (i.e. if at least one does not satisfy it), it is *invalid*.
- A wff is *satisfiable* if there is at least one interpretation that satisfies it. Otherwise (i.e. if no interpretation satisfies it), it is *unsatisfiable*.

Note how valid and satisfiable overlap: every valid wff is satisfiable, but not every satisfiable wff is valid. Similarly, every unsatisfiable wff is invalid, but not every invalid wff is unsatisfiable.

- To determine which of this terminology applies to a wff, W, you would need to consider $\mathcal{I}(W)$ for a number of different interpretations $\mathcal{I}$:

  To determine validity, invalidity, satisfiability, unsatisfiability, draw up a truth table.

  E.g.

  Is $(p_1 \wedge p_2) \Leftrightarrow (p_1 \vee p_2)$ valid or invalid? Is it satisfiable or unsatisfiable?

  Is $p \vee \neg p$ valid or invalid? Is it satisfiable or unsatisfiable?

  Is $p \wedge \neg p$ valid or invalid? Is it satisfiable or unsatisfiable?

  Question: do you always need to draw up all 2^n rows of the truth table to answer questions such as those given above?

4.2 Logical Consequence

- We're now ready to formalise the idea of argument that we introduced at the start of this material. We use the notion of *logical consequence*. (Some books refer to *logical implication* and *entailment*.)

- A wff W (the conclusion) is a *logical consequence* of a set of wffs A (the premisses) if and only if for *every* interpretation in which *all* the premisses are true, the conclusion is also true.

- In other words, whenever all the wffs in A are true, then W must also be true.

- This definition doesn't say anything about interpretations under which one or more of the wffs in A are false. In this case, we don't care whether W is true or false.

- Example: q is a logical consequence of $\{p \Rightarrow q, p\}$.

  Since, we need to consider several interpretations, we need a truth table.

p	q	$p \Rightarrow q$	p	q
true	**true**	**true**	**true**	**true**
true	**false**	**false**	**true**	**false**
false	**true**	**true**	**false**	**true**
false	**false**	**true**	**false**	**false**

  Look for all the interpretations where each wff in the set of premisses is true. There's only one in this example: the first row. Now, since the conclusion is also true under that interpretation, we can conclude that q *is* a logical consequence of $\{p \Rightarrow q, p\}$.

- Since writing 'W is a logical consequence of A' is too much effort, we use some notation: Notation:

$$A \models W$$

  E.g.

$$\{p \Rightarrow q, p\} \models q$$

  $\models$ is called the 'semantic turnstile'.

- We'll do these in the lecture.

  Show that $\{\neg p, \neg q\} \models \neg(p \vee q)$

p	q	$\neg p$	$\neg q$	$\neg(p \vee q)$
true	**true**			
true	**false**			
false	**true**			
false	**false**			

Show that $\{p \vee q, q\} \models (\neg p \wedge q) \vee p$

p	q	$p \vee q$	q	$(\neg p \wedge q) \vee p$
true	**true**			
true	**false**			
false	**true**			
false	**false**			

Show that $\{p \Rightarrow q, q\} \not\models p$

p	q	$p \Rightarrow q$	q	p
true	**true**			
true	**false**			
false	**true**			
false	**false**			

4.3 Logical Equivalence

- Two wffs W_1 and W_2 are *logically equivalent* (or simply *equivalent*) if and only if $\{W_1\} \models W_2$ and $\{W_2\} \models W_1$.
- In this case, we write

$$W_1 \equiv W_2$$

- To show $W_1 \equiv W_2$:
  - draw up truth tables for W_1 and for W_2 and confirm that $\mathcal{I}(W_1)$ is the same as $\mathcal{I}(W_2)$ for each $\mathcal{I}$ i.e. they should have the same truth value in each row.
- We'll do these in the lectures.

  Show that $\neg(p \wedge q) \equiv \neg p \vee \neg q$

p	q	$\neg(p \wedge q)$	$\neg p \vee \neg q$
true	**true**		
true	**false**		
false	**true**		
false	**false**		

  Show that $(p \Rightarrow q) \equiv \neg(p \wedge \neg q)$

p	q	$(p \Rightarrow q)$	$\neg(p \wedge \neg q)$
true	**true**		
true	**false**		
false	**true**		
false	**false**		

- We can generalise equivalences to make them into 'laws'.

  There are many such 'laws', but here are included just those that might be of most use in this module.

4.4 Laws of the Algebra of Propositions

For any propositions W_1, W_2 and W_3,...

Commutativity of $\wedge$ and $\vee$

$$W_1 \wedge W_2 \equiv W_2 \wedge W_1$$
$$W_1 \vee W_2 \equiv W_2 \vee W_1$$

Associativity of $\wedge$ and $\vee$

$$(W_1 \wedge W_2) \wedge W_3 \equiv W_1 \wedge (W_2 \wedge W_3)$$
$$(W_1 \vee W_2) \vee W_3 \equiv W_1 \vee (W_2 \vee W_3)$$

For any propositions W_1 and W_2,...

Distributivity of $\wedge$ over $\vee$ and $\vee$ over $\wedge$

$$W_1 \wedge (W_2 \vee W_3) \equiv (W_1 \wedge W_2) \vee (W_1 \wedge W_3)$$
$$W_1 \vee (W_2 \wedge W_3) \equiv (W_1 \vee W_2) \wedge (W_1 \vee W_3)$$

Absorption

$$W_1 \wedge (W_1 \vee W_2) \equiv W_1$$
$$W_1 \vee (W_1 \wedge W_2) \equiv W_1$$

De Morgan's laws

$$\neg(W_1 \wedge W_2) \equiv \neg W_1 \vee \neg W_2$$
$$\neg(W_1 \vee W_2) \equiv \neg W_1 \wedge \neg W_2$$

For any proposition W,...

Idempotence of $\wedge$ and $\vee$

$$W \wedge W \equiv W$$
$$W \vee W \equiv W$$

true-false laws

To show these laws, we need proposition symbols that denote the wff that is always true and the wff that is always false. We use **True** and **False** respectively.

$$\textbf{True} \wedge W \equiv W$$
$$\textbf{False} \wedge W \equiv \textbf{False}$$
$$\textbf{True} \vee W \equiv \textbf{True}$$
$$\textbf{False} \vee W \equiv W$$

$$W \Rightarrow \textbf{True} \equiv \textbf{True}$$
$$\textbf{True} \Rightarrow W \equiv W$$
$$W \Rightarrow \textbf{False} \equiv \neg W$$
$$\textbf{False} \Rightarrow W \equiv \textbf{True}$$
$$W \Rightarrow W \equiv \textbf{True}$$

For any proposition W,...

Involution

$$\neg\neg W \equiv W$$

Complement laws

$$W \wedge \neg W \equiv \textbf{False}$$
$$W \vee \neg W \equiv \textbf{True}$$
$$\neg\textbf{True} \equiv \textbf{False}$$
$$\neg\textbf{False} \equiv \textbf{True}$$

Here are some additional laws concerning biconditionals and conditionals.

Definition of biconditional

For any propositions W_1 and W_2,

$$W_1 \Leftrightarrow W_2 \equiv (W_1 \Rightarrow W_2) \wedge (W_2 \Rightarrow W_1)$$

Definition of conditional

For any propositions W, W_1, W_2 and W_3,

$$W_1 \Rightarrow W_2 \equiv \neg W_1 \vee W_2$$

Confirm both of the above in your own time by drawing truth tables.

Since intuitions about the conditional are so tricky, let's see what we learn from the following:

W_1	W_2	Conditional $W_1 \Rightarrow W_2$	Converse $W_2 \Rightarrow W_1$
true	**true**	**true**	**true**
true	**false**	**false**	**true**
false	**true**	**true**	**false**
false	**false**	**true**	**true**

W_1	W_2	Inverse $\neg W_1 \Rightarrow \neg W_2$	Contrapositive $\neg W_2 \Rightarrow \neg W_1$
true	**true**	**true**	**true**
true	**false**	**true**	**false**
false	**true**	**false**	**true**
false	**false**	**true**	**true**

So,

Contrapositive law

For any propositions W_1 and W_1,

$$W_1 \Rightarrow W_2 \equiv \neg W_2 \Rightarrow \neg W_1$$

The truth-table above also shows that a conditional is not equivalent to its converse: you would agree that 'If it is raining, then I get wet' does not have the same truth conditions as 'If I get wet, then it is raining'.

The laws give another 'method' for showing $W_1 \equiv W_2$

- use algebraic manipulation

E.g. to show $p \vee q \equiv \neg(\neg p \wedge \neg q)$ algebraically.

$$\begin{aligned} \text{RHS} \quad &= \neg(\neg p \wedge \neg q) \\ &= \neg\neg(p \vee q) && \text{(by De Morgan's Law)} \\ &= p \vee q && \text{(by Involution)} \\ &= \text{LHS} \end{aligned}$$