Lecture 3:
The Semantics of Propositional Logic

Aims:

- To discuss what we mean by an interpretation;
- To look at the semantics of compound wffs. connectives.

3.1 The Semantics of Propositional Logic

- Wffs represent statements, and these are either true or false.

3.1.1 Atomic Wffs

- The truth values of atomic wffs (propositional symbols) are *stipulated*. In other words, we do not spend time debating whether a proposition symbol such as p,

$$p =_{\text{def}} \text{The moon is made of cheese}$$

  denotes a true proposition or a false one. Instead, I stipulate what its truth value is. (In fact, a significant part of what we do will involve considering both what happens when p denotes a true proposition and when p denotes a false proposition, so these stipulations aren't required too often anyway.)

- A stipulation of the truth values of atomic wffs is called an *interpretation* of the atomic wffs. Some books use the phrases *truth assignment* or *value assignment* or *truth valuation* instead of 'interpretation'.

- Example (in two different notations):

p has truth value **true**

q has truth value **true**

r has truth value **false**

$\mathcal{I}(p)$ is **true**

$\mathcal{I}(q)$ is **true**

$\mathcal{I}(r)$ is **false**

3.1.2 Compound Wffs

- The truth value of a compound wff is determined by the truth values of its component wffs. So, we must now look at each connective in turn and see how a wff that is formed using that connective receives its truth value from the truth values of the subwffs of the compound.

3.1.3 Negation

- Given any wff W, another wff, called the *negation* of W, can be formed, and is denoted

$$\neg W$$

This is read as 'not W'. Some books use

$$\sim W \qquad W' \qquad \overline{W} \qquad \textbf{not } W$$

- For any wff W,

$$\begin{array}{ll} \mathcal{I}(\neg W) =_{\text{def}} & \text{if } \mathcal{I}(W) \text{ is } T \\ & \quad \text{then } F \\ & \quad \text{else } T \end{array}$$

i.e. to determine the truth value of $\neg W$, you must determine the truth value of W, i.e. $\mathcal{I}(W)$. Then, $\mathcal{I}(\neg W)$ is **true** if $\mathcal{I}(W)$ is **false** and $\mathcal{I}(\neg W)$ is **false** if $\mathcal{I}(W)$ is **true**.

The above is a perfectly good specification of the semantics of $\neg W$, but a tabular representation is more often used, as it is clearer:

W	$\neg W$
true	**false**
false	**true**

You can see that this says the same thing as we said above.

- Example

$$\begin{array}{lll} p_1 & =_{\text{def}} & \text{Paris is in France} \\ p_2 & =_{\text{def}} & 2+2=5 \\ p_3 & =_{\text{def}} & \text{Clyde is intelligent} \end{array}$$

Consider the interpretation

$$\mathcal{I}(p_1) \text{ is } \textbf{true}; \qquad \mathcal{I}(p_2) \text{ is } \textbf{false}; \qquad \mathcal{I}(p_3) \text{ is } \textbf{true}$$

then

$$\mathcal{I}(\neg p_1) \text{ is } \textbf{false}; \qquad \mathcal{I}(\neg p_2) \text{ is } \textbf{true}; \qquad \mathcal{I}(\neg p_3) \text{ is } \textbf{false}$$

Class Exercise

- Consider

$$\begin{array}{lll} q_1 & =_{\text{def}} & \text{Clyde likes Flopsy} \\ q_2 & =_{\text{def}} & \text{Clyde is happy} \end{array}$$

1. Paraphrase into English

$$\neg q_1$$

2. Translate into propositional logic

   It's not the case that Clyde is happy.
   Clyde isn't happy.
   Clyde is unhappy.
   Clyde is sad.

3.1.4 Conjunction

- Any two wffs W_1 and W_2 can be combined to form a compound wff, called the *conjunction* of W_1 and W_2, and denoted

  $$W_1 \wedge W_2$$

  W_1 and W_2 are called the *conjuncts*. Some books use

  $$W_1.W_2 \qquad W_1\&W_2 \qquad W_1 \textbf{ and } W_2 \qquad W_1W_2$$

- For any wffs W_1 and W_2,

  $\mathcal{I}(W_1 \wedge W_2) =_{\text{def}}$ if $\mathcal{I}(W_1)$ is **true** and $\mathcal{I}(W_2)$ is **true**
  then **true**
  else **false**

  Again, the tabular presentation is clearer, so this is what you should learn:

W_1	W_2	$W_1 \wedge W_2$
true	**true**	**true**
true	**false**	**false**
false	**true**	**false**
false	**false**	**false**

  Note that we need four rows to capture all the possible combinations of truth values for W_1 and W_2.

- Example

  $p_1 =_{\text{def}}$ Paris is in France
  $p_2 =_{\text{def}}$ Cork is in Ireland
  $p_3 =_{\text{def}}$ $2 + 2 = 5$

  Consider

  $\mathcal{I}(p_1)$ is **true**; $\mathcal{I}(p_2)$ is **true**; $\mathcal{I}(p_3)$ is **false**

  then, e.g.,

  $\mathcal{I}(p_1 \wedge p_2)$ is **true**; $\mathcal{I}(p_1 \wedge p_3)$ is **false**; $\mathcal{I}(p_3 \wedge p_3)$ is **false**

- The logical behaviour of $\wedge$ is in some accord with that of the English word 'and' (when used to join two sentences):

  $p_1 \wedge p_3$ 'Paris is in France and $2 + 2 = 5$'
  'Both, Paris is in France and $2 + 2 = 5$'

Class Exercise

- Consider

$$\begin{array}{lll} q_1 & =_{\text{def}} & \text{Clyde likes Flopsy} \\ q_2 & =_{\text{def}} & \text{Clyde is happy} \end{array}$$

1. Paraphrase into English

$$q_1 \wedge \neg q_2$$

2. Translate into propositional logic

   It's not the case that both Clyde likes Flopsy and Clyde is happy.

3.1.5 Disjunction

- Any two wffs W_1 and W_2 can be combined to form a compound wff, called the *disjunction* of W_1 and W_2, and denoted

$$W_1 \vee W_2$$

This is read as 'W_1 or W_2' or 'W_1 or W_2 or both' or 'W_1 inclusive-or W_2'. W_1 and W_2 are called the *disjuncts*. Some books use

$$W_1 + W_2 \qquad W_1 \textbf{ or } W_2$$

- For any wffs W_1 and W_2,

$$\begin{array}{ll} \mathcal{I}(W_1 \vee W_2) =_{\text{def}} & \text{if } \mathcal{I}(W_1) \text{ is } \textbf{true} \text{ or } \mathcal{I}(W_2) \text{ is } \textbf{true} \text{ (or both)} \\ & \text{then } \textbf{true} \\ & \text{else } \textbf{false} \end{array}$$

More simply...

W_1	W_2	$W_1 \vee W_2$
true	**true**	**true**
true	**false**	**true**
false	**true**	**true**
false	**false**	**false**

- Example

$$\begin{array}{lll} p_1 & =_{\text{def}} & \text{Paris is in France} \\ p_2 & =_{\text{def}} & \text{Cork is in Ireland} \\ p_3 & =_{\text{def}} & 2 + 2 = 5 \end{array}$$

Consider

$$\mathcal{I}(p_1) \text{ is } \textbf{true}; \qquad \mathcal{I}(p_2) \text{ is } \textbf{false}; \qquad \mathcal{I}(p_3) \text{ is } \textbf{false}$$

"Hold on!" you cry. "p_2 is **true** in our world!"

"I know," I reply. "I just wanted to emphasise the point that *I stipulate* the truth values and you have to go along with them."

then, e.g.

$$\mathcal{I}(p_1 \vee p_2) \text{ is } \textbf{true}; \qquad \mathcal{I}(p_2 \vee p_3) \text{ is } \textbf{false}; \qquad \mathcal{I}(p_1 \vee p_1) \text{ is } \textbf{true}$$

- The logical behaviour of $\vee$ is in some accord with that of the English word 'or' (when used to join two sentences):

$$p_1 \lor p_3 \quad \text{‘Paris is in France or } 2+2=5 \text{ (or both)’}$$

- The English word ‘or’ has at least two distinct uses:

‘W_1 or W_2 or both’	*inclusive-or*
‘W_1 or W_2 but not both’	*exclusive-or*

  Inclusive-or is the one we have introduced already.

  Exclusive-or is also truth-functional, so we could have a connective for it. $\oplus$ is the symbol that is often used. We would give this connective a semantics just like that of $\lor$ except that in the first row of the table, where both $\mathcal{I}(W_1)$ and $\mathcal{I}(W_2)$ are **true**, $\mathcal{I}(W_1 \oplus W_2)$ would be **false**, not **true**. Unless otherwise stated, we are using inclusive-or in this module. Exclusive-or can be captured using $\lor, \land$ and $\neg$ together (see below).

  We see again the advantage of using a formal language in making clear an ambiguity in English. Of course, this means that we need to be extra careful when translating sentences of English into logic: should we use $\lor$ or $\oplus$? Generally, $\lor$ will do. However, sentences that begin with the word ‘either’ often signal use of $\oplus$.

$p_1 \lor p_2$	‘Paris is in France or Cork is in Ireland (or both)’
$(p_1 \lor p_2) \land \neg(p_1 \land p_2)$	‘Paris is in France or Cork is in Ireland (but not both)’ ‘*Either* Paris is in France or Cork is in Ireland’

3.1.6 Conditional

- Any two wffs W_1 and W_2 can be combined to form a compound wff denoted

$$W_1 \Rightarrow W_2$$

  called a *conditional*. This can be read as ‘if W_1 then W_2’. W_1 is called the *antecedent* and W_2 is called the *consequent*. Some books use

$$W_1 \rightarrow W_2 \qquad W_1 \supset W_2$$

  instead of $W_1 \Rightarrow W_2$.

  Note also that many books call this connective *implication* or *material implication*, instead of the ‘conditional’. I try to avoid this because it encourages students to ignore the formal semantics (below) and try to ‘get by’ using informal paraphrases into English based on the word ‘implies’, resulting in them being led astray by this somewhat inappropriate paraphrase.

  However, using the word ‘conditional’ brings about another possible confusion because we also use this word when referring to **if** commands in programming languages. I assume that you will be able to realise that there are two different concepts here: the one we are referring to will be clear from the context.

- For any wffs W_1 and W_2,

$$\mathcal{I}(W_1 \Rightarrow W_2) =_{\text{def}} \begin{array}{l}\text{if } \mathcal{I}(W_1) \text{ is } \textbf{false} \text{ or } (W_2) \text{ is } \textbf{true} \\ \text{then } \textbf{true} \\ \text{else } \textbf{false}\end{array}$$

  More simply. . .

W_1	W_2	$W_1 \Rightarrow W_2$
true	**true**	**true**
true	**false**	**false**
false	**true**	**true**
false	**false**	**true**

- I shall attempt an explanation of why this connective is thought to do something like the job of English 'if... then'. Consider

$$\begin{array}{ll} p & =_{\text{def}} \text{I win the election} \\ q & =_{\text{def}} \text{Taxes will fall} \end{array}$$

A politician utters 'If I win the election, then taxes will fall', i.e. $p \Rightarrow q$. Under what circumstances would this be a lie?

  - Suppose s/he wins and taxes fall:

$$\mathcal{I}(p) \text{ is } \textbf{true}; \qquad \mathcal{I}(q) \text{ is } \textbf{true}; \qquad \text{so } \mathcal{I}(p \Rightarrow q) \text{ is } \textbf{true}$$

    The semantics says s/he was truthful. Quite right.

  - Suppose s/he wins and taxes rise or stay the same:

$$\mathcal{I}(p) \text{ is } \textbf{true}; \qquad \mathcal{I}(q) \text{ is } \textbf{false}; \qquad \text{so } \mathcal{I}(p \Rightarrow q) \text{ is } \textbf{false}$$

    The semantics says s/he lied in this case. Quite right too.

  - Suppose s/he loses and taxes fall:

$$\mathcal{I}(p) \text{ is } \textbf{false}; \qquad \mathcal{I}(q) \text{ is } \textbf{true}; \qquad \text{so } \mathcal{I}(p \Rightarrow q) \text{ is } \textbf{true}$$

    The semantics says s/he was truthful. See below.

  - Suppose s/he loses and taxes rise or stay the same:

$$\mathcal{I}(p) \text{ is } \textbf{false}; \qquad \mathcal{I}(q) \text{ is } \textbf{false}; \qquad \text{so } \mathcal{I}(p \Rightarrow q) \text{ is } \textbf{true}$$

    The semantics says s/he was truthful. See below.

In these last two cases, it would seem very harsh to say that s/he lied, i.e. that $\mathcal{I}(p \Rightarrow q)$ is **false**. Of course, it also feels a bit bizarre to say that $\mathcal{I}(p \Rightarrow q)$ is **true**. But every wff is either **true** or **false**. So we need a convention and the convention is that $\mathcal{I}(p \Rightarrow q)$ is **true** in both these cases.

3.1.7 Biconditional

- Any two wffs W_1 and W_2 can be combined to form a compound wff, denoted

$$W_1 \Leftrightarrow W_2$$

which is called a *biconditional*. This is read as 'W_1 if and only if W_2'. Some books use

$$W_1 \leftrightarrow W_2 \qquad W_1 \equiv W_2$$

instead of $W_1 \Leftrightarrow W_2$. We'll need the symbol $\equiv$ as part of our metalanguage later, so we don't want to use it for something else here.

Some books call this connective the 'equivalence' connective but that risks confusion with the metalanguage too (as we shall see). And others call it *bi-implication*. But I'm avoiding the word 'implies' altogether.

- For any wffs W_1 and W_2,
  $\mathcal{I}(W_1 \Leftrightarrow W_2) =_{\text{def}}$ if $\mathcal{I}(W_1)$ is the same as $\mathcal{I}(W_2)$
  then **true**
  else **false**

  More simply...

W_1	W_2	$W_1 \Leftrightarrow W_2$
true	**true**	**true**
true	**false**	**false**
false	**true**	**false**
false	**false**	**true**

- Example

$$\begin{array}{lll} p_1 & =_{\text{def}} & \text{Paris is in France} \\ p_2 & =_{\text{def}} & \text{Cork is in Ireland} \\ p_3 & =_{\text{def}} & 2+2=5 \end{array}$$

  Consider

  $\mathcal{I}(p_1)$ is **true**; $\mathcal{I}(p_2)$ is **true**; $\mathcal{I}(p_3)$ is **false**

  then, e.g.,

  $\mathcal{I}(p_1 \Leftrightarrow p_2)$ is **true**; $\mathcal{I}(p_1 \Leftrightarrow p_3)$ is **false**; $\mathcal{I}(p_3 \Leftrightarrow p_3)$ is **true**

- The following show possible English words and phrases that might translate as the biconditional.

  $p_1 \Leftrightarrow p_2$ 'Paris is in France if and only if Cork is in Ireland'
  'Paris being in France is a necessary and sufficient condition for Cork to be in Ireland'

- In maths textbooks, the English phrase 'if and only if', which sees fairly regular use in the metalanguage, is often abbreviated to 'iff'.

3.2 Truth Tables

- Let's find the truth-values of a few complex wffs. Given this interpretation:

  $\mathcal{I}(p_1)$ is **true**; $\mathcal{I}(p_2)$ is **false**; $\mathcal{I}(p_3)$ is **true**

  What is $\mathcal{I}(p_1 \wedge (p_2 \Rightarrow p_3))$?

  Parentheses determine that the first subwff to be evaluated is $p_2 \Rightarrow p_3$. Because $\mathcal{I}(p_2)$ is **false** and $\mathcal{I}(p_3)$ is **true**,

  $\mathcal{I}(p_2 \Rightarrow p_3)$ is **true**

  from row three of the table defining the semantics of $\Rightarrow$.

  And now we know that $\mathcal{I}(p_1)$ is **true** and $\mathcal{I}(p_2 \Rightarrow p_3)$ is **true**, we can determine that

  $\mathcal{I}(p_1 \wedge (p_2 \Rightarrow p_3))$ is **true**

  from the first row of the table defining the semantics of $\wedge$.

  So the whole wff is true (*for this interpretation of the atomic wffs*): this wff is true *under* this interpretation.

- Given a wff, W, and an interpretation of the wff's atomic wffs, $\mathcal{I}$, we say that $\mathcal{I}$ *satisfies* W if and only if $\mathcal{I}(W)$ is **true** i.e. if the wff is true under that interpretation then that interpretation satisfies that wff.

  Hence, $\mathcal{I}$ from above satisfies $p_1 \wedge (p_2 \Rightarrow p_3)$.

- Let's do it all again for a different interpretation of the atomic wffs:

$$\mathcal{I}(p_1) \text{ is } \textbf{true}; \qquad \mathcal{I}(p_2) \text{ is } \textbf{true}; \qquad \mathcal{I}(p_3) \text{ is } \textbf{false}$$

  This time, we'll set things out in a tabular fashion, which results in a tidier presentation.

  We draw a table with a column for each subwff of the wff:

p_1	p_2	p_3	$(p_2 \Rightarrow p_3)$	$p_1 \wedge (p_2 \Rightarrow p_3)$

  Then we can fill in the truth values for the atomic wffs (using our new interpretation $\mathcal{I}$ from above):

p_1	p_2	p_3	$(p_2 \Rightarrow p_3)$	$p_1 \wedge (p_2 \Rightarrow p_3)$
true	**true**	**false**		

  Now we fill in the next column with reference to the semantics of $\Rightarrow$. This time, the second row of the table that defines the semantics of $\Rightarrow$ applies:

p_1	p_2	p_3	$(p_2 \Rightarrow p_3)$	$p_1 \wedge (p_2 \Rightarrow p_3)$
true	**true**	**false**	**false**	

  And now we can fill in the final column using the semantics of $\wedge$. Since $\mathcal{I}(p_1)$ is **true** and $\mathcal{I}(p_2 \Rightarrow p_3)$ is **false** (from the third column above), $\mathcal{I}(p_1 \wedge (p_2 \Rightarrow p_3))$ is **false** (second row of the table defining the semantics of $\wedge$):

p_1	p_2	p_3	$(p_2 \Rightarrow p_3)$	p_1	$\wedge$	$(p_2 \Rightarrow p_3)$
true	**true**	**false**	**false**		**false**	
					$\triangle$	

  Obviously, the fourth column here is now not really important. It was just part of our 'working'.

  So the whole wff is false under this interpretation, i.e. this interpretation does not satisfy $p_1 \wedge (p_2 \Rightarrow p_3)$.

- We'll do it yet again, with yet another interpretation:

$$\mathcal{I}(p_1) \text{ is } \textbf{false}; \qquad \mathcal{I}(p_2) \text{ is } \textbf{false}; \qquad \mathcal{I}(p_3) \text{ is } \textbf{false}$$

  And we'll illustrate another way of doing the 'working'.

  We draw a table as follows:

p_1	p_2	p_3	p_1	$\wedge$	$(p_2$	$\Rightarrow$	$p_3)$

  We use the interpretation above to fill in the truth values for the atomic wffs:

p_1	p_2	p_3	p_1	$\wedge$	$(p_2$	$\Rightarrow$	$p_3)$
false	**false**	**false**					

Then we copy these values to columns headed by occurrences of those wffs in the compound wff:

p_1	p_2	p_3	p_1	$\wedge$	$(p_2$	$\Rightarrow$	$p_3)$
false	**false**	**false**	**false**		**false**		**false**

Now, since parentheses dictate that the $\Rightarrow$ is evaluated first, we work out $\mathcal{I}(p_2 \Rightarrow p_3)$ (which, given that $\mathcal{I}(p_2)$ is **false** and $\mathcal{I}(p_3)$ is **false**, is **true**), and we place this into the column headed by the $\Rightarrow$:

p_1	p_2	p_3	p_1	$\wedge$	$(p_2$	$\Rightarrow$	$p_3)$
false	**false**	**false**	**false**		**false**	**true**	**false**

And now we evaluate the conjunction $\mathcal{I}(p_1 \wedge (p_2 \Rightarrow p_3)$ (which, given that we now know that $\mathcal{I}(p_1)$ is **false** and $\mathcal{I}(p_2 \Rightarrow p_3)$ is **true**, is **false**), and we write this into the column headed by $\wedge$:

p_1	p_2	p_3	p_1	$\wedge$	$(p_2$	$\Rightarrow$	$p_3)$
false	**false**	**false**	**false**	**false**	**false**	**true**	**false**

Obviously, the fourth, sixth, seventh and eighth columns here are not really important: they're just 'working'. The 'result' is in the fifth column.

So, the whole wff is false under this interpretation, i.e. this interpretation does not satisfy $p_1 \wedge (p_2 \Rightarrow p_3)$.

- Most often we need to compute the truth value of a wff under *every possible* interpretation of its atomic wffs. The tabular presentation above can be extended to do a neat job of this: we use one row for each interpretation.
- If there are n different proposition symbols in a compound wff, then there are 2^n different interpretations of those atomic wffs.
- The wff $p_1 \wedge (p_2 \Rightarrow p_3)$ has three different proposition symbols, so there are $2^3 = 8$ possible interpretations, hence our truth table will have 8 rows.

p_1	p_2	p_3	p_1	$\wedge$	$(p_2 \Rightarrow p_3)$
true	**true**	**true**		**true**	
true	**true**	**false**		**false**	
true	**false**	**true**		**true**	
true	**false**	**false**		**true**	
false	**true**	**true**		**false**	
false	**true**	**false**		**false**	
false	**false**	**true**		**false**	
false	**false**	**false**		**false**	

Notes:

1. The third, second and eighth rows are the three interpretations we looked at previously.
2. In the lectures I will show you a systematic way to make sure that none of the 2^n rows is forgotten.
3. In the truth table above, I haven't shown any of my 'working'. I'll show this in the lecture.

Class Exercises

- We will complete these truth tables during the lecture if there is time; otherwise, you can do them yourself in your own time.

p_1	p_2	$(p_1$	$\wedge$	$p_2)$	$\Leftrightarrow$	$(p_1$	$\vee$	$p_2)$
true	**true**							
true	**false**							
false	**true**							
false	**false**							

p	p	$\vee$	$\neg$	p
true				
false				

p	p	$\wedge$	$\neg$	p
true				
false				