Fuzzy Logic

1 Uncertainty and Vagueness

Uncertainty and vagueness are major challenges in AI. Uncertainty is about *degree of belief* and it can be handled mathematically by *probability*. Vagueness is about *degree of truth* and it can be handled mathematically by *fuzzy logic*.

Our reactive agents (and intelligent agents in general) must be able to deal with both uncertainty and vagueness. For example, one major source of uncertainty is the unreliability of the sensing equipment. However, we'll look now at vagueness and fuzzy logic. Fuzzy logic has been applied with some considerable success to reactive control tasks.

2 Some Motivation: Fuzzy Control

Consider an anti-locking brake system, controlled by a programmable controller. The operation of such a system is akin to the sense/plan/act cycle. Readings are taken of, e.g., brake temperature and vehicle speed. An action, e.g. a change in brake pressure, is then selected and executed. The cycle then repeats. The aim is to maintain some system state or to smoothly change the state of the system in response to changes in the environment.

Traditional control systems typically use differential equations to define responses to sensor values. But this approach has problems:

- In some cases, solving the complex equations can be too computationally expensive for real-time control;
- In some cases, the effect is a 'bang-bang' control regime: actions are repeatedly too drastic and cause overshooting of desired outcomes;
- In some cases, e.g. for new or ill-understood devices & environments, the equations may not be known yet; developing them might be years of work; the resulting equations might be difficult to understand.

An alternative is to use a system of fuzzy rules, such as the following:

if brake temperature is warm $\wedge$ vehicle speed is not very fast **then** brake pressure is slightly decreased

which uses imprecise terms in its condition and action.

Such rules can be cheap to apply, and have been found to give quite smooth control. Most importantly, fuzzy controllers can often be quickly developed. The rules can come from interviewing experts, looking at their manuals, or observing their actions. Given enough data, the rules can be automatically evolved or learned. The rules are readily understandable, and therefore more maintainable.

Hybrid approaches, where fuzzy control improves an existing equation-based controller, are also common.

Note that, while our current interest in fuzzy logic is its application in fuzzy control, it has been used for a variety of purposes throughout AI (although its use is always controversial).

3 Fuzzy Set Theory

Fuzzy control uses fuzzy logic, and fuzzy logic is built on fuzzy set theory. Traditional, 'non-fuzzy' sets are sometimes referred to as *crisp sets* to distinguish them from *fuzzy sets*.

In (crisp) set theory, we have a *universe of discourse*, U: the collection of all possible objects under consideration. A (crisp) set, A, is then a collection of objects, drawn from the universe of discourse. Each object in U is either in A or not in A. More formally, for every set A there is a *membership function*, f_A, which, for every object a in U returns **true** or **false** according to whether a is in A or not.

Example Let U be a collection of students; call them a, b, c, d, and e. Let set $C = \{a, b, d\}$ be the set of these students who are studying Computer Science. The membership function for CS, f_C, is as follows:

$$\begin{aligned} f_C(a) &= \textbf{true} \\ f_C(b) &= \textbf{true} \\ f_C(c) &= \textbf{false} \\ f_C(d) &= \textbf{true} \\ f_C(e) &= \textbf{false} \end{aligned}$$

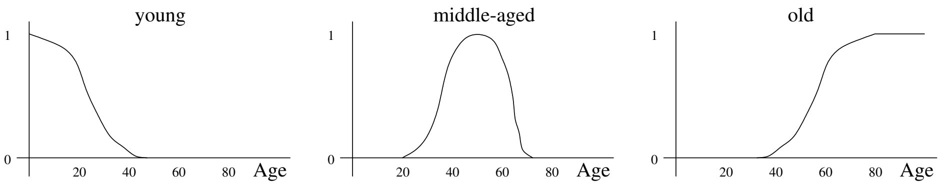
The idea in fuzzy set theory is that set membership should be a matter of degree, not simply **true** or **false** . The key generalisation is to allow the membership function to return not simply **true** or **false** , but a grade of membership, usually denoted by a real number between 0 (absolutely not a member) and 1 (absolutely a member).

Example The membership function for the fuzzy set T of tall students might be, e.g.:

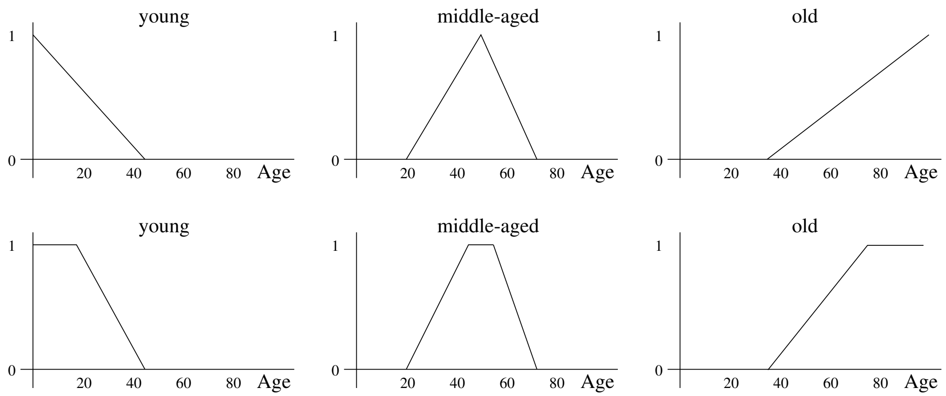
$$\begin{aligned} f_T(a) &= 0.2 \\ f_T(b) &= 0.8 \\ f_T(c) &= 0.0 \\ f_T(d) &= 1.0 \\ f_T(e) &= 0.5 \end{aligned}$$

This shows that d is definitely tall and c definitely is not. a, b and e have some claims to tallness, e.g. it is half-true to say that e is tall.

Membership functions for fuzzy sets (especially when the set is concerned with ages, heights or other (continuous) numeric measurements) are often shown graphically. For example, here are plots of possible membership functions for person ages:



We see that the exact shape may vary: it might be bell-shaped, s-shaped or reverse s-shaped. In fact, it is often convenient computationally to use triangular- or trapezoidal-shaped functions:



Question *What, in fuzzy set theory, would the membership function of the empty set look like?*

The definitions of set operations (such as union) need to be extended to cope with fuzzy sets. Let A, A_1 and A_2 be fuzzy sets drawn from universe U.

Union $A_1 \cup A_2$ For crisp sets, the union of A_1 and A_2 is the set of elements that are either in A_1 or in A_2. For fuzzy sets, the membership function for the union of A_1 and A_2, $f_{A_1 \cup A_2}$, is defined by

$$f_{A_1 \cup f_2}(a) \triangleq \max(f_{A_1}(a), f_{A_2}(a))$$

Intersection $A_1 \cap A_2$ For crisp sets, the intersection of A_1 and A_2 is the set of elements that are in both A_1 and A_2. For fuzzy sets, the membership function for the intersection of A_1 and A_2, $f_{A_1 \cap A_2}$, is defined by

$$f_{A_1 \cap A_2}(a) \triangleq \min(f_{A_1}(a), f_{A_2}(a))$$

Complement A' For crisp sets, the complement of A is the set of elements that are in the universe of discourse but are not in A. For fuzzy sets, the membership function for the complement of A, $f_{A'}$, is defined by

$$f_{A'}(a) = 1 - f_A(a)$$

Exercise *Draw the graph of the membership function for the set that is the union of the sets of young ages and middle-aged ages. Draw the graph for their intersection too.*

There is a whole new set of operations (ones that don't exist in crisp set theory) called *modifiers*. These have approximately the same effect as words and phrases such as *"very"*, *"more or less"*, etc. We want the very operator to have an intensifying effect and the morl (more or less) operator to reduce the intensity. Here are possible definitions:

$$f_{\mathrm{very}\ A}(a) \triangleq (f_A(a))^2$$

$$f_{\mathrm{morl}\ A}(a) \triangleq (f_A(a))^{\frac{1}{2}}$$

Exercise *Draw the graphs for the set of very young person ages and the set of more or less young person ages.*

Much more could be said on this subject (including its extension to fuzzy relations) but we move on now to fuzzy logic.

4 Fuzzy Logic

In classical logic, a statement, W, is either **true** or **false** . Again, we can use functions to formalise this. The function we need is called an *interpretation function*, $\mathcal{I}$. If p represents the statement 'the ball is red', then $\mathcal{I}(p)$ will return either **true** of **false** .

In fuzzy logic, $\mathcal{I}$ can return degrees of truth, typically in the [0, 1] interval. For example, $\mathcal{I}(p)$ could be 0.6 (perhaps if the ball is a reddish-orange colour).

The following connectives are defined in a way that is based on the set theory operations:

$$\begin{aligned} \mathcal{I}(W_1 \vee W_2) &\triangleq \max(\mathcal{I}(W_1), \mathcal{I}(W_2)) \\ \mathcal{I}(W_1 \wedge W_2) &\triangleq \min(\mathcal{I}(W_1), \mathcal{I}(W_2)) \\ \mathcal{I}(\neg W) &\triangleq 1 - \mathcal{I}(W) \end{aligned}$$

There are problems in defining the conditional $W_1 \Rightarrow W_2$ (and hence also the biconditional, $W_1 \Leftrightarrow W_2$). Around 72 alternative definitions have been proposed. One is

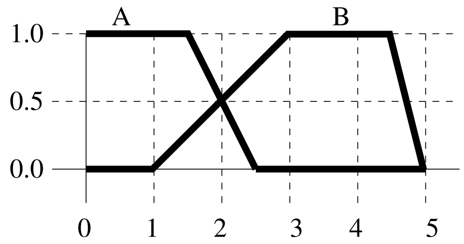
$$\mathcal{I}(W_1 \Rightarrow W_2) \triangleq \begin{cases} 1 & \text{if } \mathcal{I}(W_1) \leq \mathcal{I}(W_2) \\ \mathcal{I}(W_2) & \text{otherwise} \end{cases}$$

In classical logic, $W_1 \Rightarrow W_2 \equiv \neg W_1 \vee W_2$. But, this equivalence does not hold when using the above definition of the conditional in fuzzy logic.

We needn't get too worried about these problems with defining the conditional. In fuzzy control (next lecture), we will only need conjunction ($\wedge$), disjunction ($\vee$) and negation ($\neg$). We will be writing condition-action rules like the ones we used in production systems, but now the conditions and actions may be fuzzy.

Exercises

1. Suppose U, the universe of discourse is $\{0, 1, 2, 3, 4, 5\}$. The graph below shows the membership function for the fuzzy sets A and B:



Draw separate graphs that show the membership functions for the following fuzzy sets:

(a) A'

(b) B'

(c) $A \cup B$

(d) $A \cap B$

(e) $(A \cup B)'$

(f) $(A \cap B)'$

2. We have three statements of fuzzy logic, p, q and r. Here are their degrees of truth:

$$\mathcal{I}(p) = 0.2; \mathcal{I}(q) = 0.5; \mathcal{I}(r) = 0.7$$

Compute the degrees of truth of the following:

(a) $p \wedge q$

(b) $\neg p \vee \neg q$

(c) $\neg(p \wedge q) \vee r$

(d) $p \Rightarrow q$ (Of the 72 definitions, use the one I gave above!)

(e) $\neg p \vee q$ (Note how the answer to this one is different from the answer to the previous one, which shows that in fuzzy logic $W_1 \Rightarrow W_2$ is not always equal to $\neg W_1 \vee W_2$.)

(f) $(p \Rightarrow q) \Rightarrow r$