The Semantics of FOPL

1 Introduction

We have to say what the expressions of FOPL mean. In FOPL, the meaning of the lowest level expressions (constant symbols, function symbols and predicate symbols) will be *stipulated* by an *interpretation*. An interpretation is simply a mapping from symbols to the things they stand for. Then, the meaning of larger expressions, including wffs, will be determined by composing the meanings of their subexpressions.

It's tempting to think that all this semantic apparatus isn't necessary. We might think this for two reasons:

- It doesn't seem to be necessary for natural languages, such as English, so why is it necessary for formal languages such as FOPL?

  But actually it *is* necessary for natural languages. At some point, we did have to learn/be told how to map the symbols of English (the words) to the things they stand for. But this happened in our childhood, so we no longer recall going through this process.

- We also tend to think that the meaning of wffs is 'obvious':

  $$\textit{likes}(\textit{clyde}, \textit{gertie})$$

  $$\forall x\, \textit{likes}(x, \textit{clyde})$$

  — Clyde likes Gertie and everybody likes Clyde, of course!

  But this is only because we often choose predicate, function and constant symbols that resemble English words (e.g. *likes*, *clyde* and *gertie* above). But this is dangerous! How can you be sure what the author of the wffs above intended. Does $\textit{likes}(\textit{clyde}, \textit{gertie})$ correspond to "Clyde likes Gertie" or "Gertie likes Clyde" or something utterly counter-intuitive such as "Clyde borrows a fiver from Gertie", ... And what if the author doesn't use symbols that resemble English words at all, e.g. $p(a, b) \vee p(a, c)$ — what does this mean?

2 The Universe of Discourse

We assume that there is some set of entities (possibly an infinite set) which we refer to as the *universe of discourse*, and which we denote as $\mathcal{U}$.

It is obviously possible to define a large number of relations and functions *on* $\mathcal{U}$.

We'll look at an example universe of discourse which I call The Books World.

- Entities: Here we have five entities (excluding the table): Down&Out, Zen, Cuckoo, Jude and Spire.

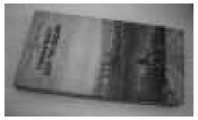
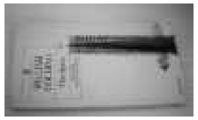
- Functions: Here we have such functions as the unary function that maps from a book to the book directly on top of it.
- Relations: Here we have such relations as the binary relation that tells us whether a pair of books is such that the second is *somewhere* above the first, and the unary relation that tells us which books have a clear top.

3 Interpretations

In FOPL, an interpretation $\mathcal{I}$ makes the following mappings:

1. For every constant symbol C, $\mathcal{I}(C)$ is an entity from $\mathcal{U}$. (Different constant symbols need not denote different entities, and there may be unnamed entities.)
2. For every n-ary function symbol F, $\mathcal{I}(F)$ is an n-ary function on $\mathcal{U}$, i.e. one that maps n-tuples to an entity. (Different function symbols need not denote different functions, and there may be unnamed functions.)
3. For every n-ary predicate symbol P, $\mathcal{I}(P)$ is an n-ary relation on $\mathcal{U}$, i.e. one that holds of n-tuples. (Different predicate symbols need not denote different relations, and there may be unnamed relations.)

Let's look at an example of an interpretation function. We'll assume we're going to use constant symbols a, b, c, d, e, a unary function symbol f and the binary predicate symbols p and q. And we'll map these symbols to aspects of The Books World. We'll call the interpretation $\mathcal{I}_1$.

$\mathcal{I}_1(a)$ is		(Down&Out)
$\mathcal{I}_1(b)$ is		(Zen)
$\mathcal{I}_1(c)$ is		(Cuckoo)
$\mathcal{I}_1(d)$ is		(Jude)
$\mathcal{I}_1(e)$ is		(Spire)

$\mathcal{I}_1(f)$ is	the unary function that maps a book to the book *directly on top* of it, i.e. the function whose maplets are {ZEN $\mapsto$ Down&Out, Cuckoo $\mapsto$ Zen, Spire $\mapsto$ Jude}
$\mathcal{I}_1(p)$ is	the binary relation whose extension contains pairs of books where the second book of the pair is *directly on top* of the first book of the pair, i.e. its tuples are $\{\langle$Zen, Down&Out$\rangle$, $\langle$Cuckoo, Zen$\rangle$, $\langle$Spire, Jude$\rangle\}$
$\mathcal{I}_1(q)$ is	the binary relation whose extension contains pairs of books where the second book of the pair is *somewhere above* the first book of the pair, i.e. its tuples are $\{\langle$Zen, Down&Out$\rangle$, $\langle$Cuckoo, Zen$\rangle$, $\langle$Cuckoo, Down&Out$\rangle$, $\langle$Spire, Jude$\rangle\}$

$\mathcal{I}_1$ is, however, only one possible interpretation (one possible mapping from our symbols to our universe). Numerous other interpretations are also possible. For example, here is another example interpretation, $\mathcal{I}_2$, for the same universe of discourse and the same symbols.

$\mathcal{I}_2$ is identical to $\mathcal{I}_1$ except:

$\mathcal{I}_2(a)$ is 

(Zen)

$\mathcal{I}_2(b)$ is 

(Down&Out)

$\mathcal{I}_2(p)$ is	the binary relation whose extension contains pairs of books where the second book of the pair is *directly below* the first book of the pair, i.e. its tuples are $\{\langle$Down&Out, Zen$\rangle$, $\langle$Zen, Cuckoo$\rangle$, $\langle$Jude, Spire$\rangle\}$
$\mathcal{I}_2(q)$ is	the binary relation whose extension contains pairs of books where the second book of the pair is *directly on top* of the first book of the pair, i.e. its tuples are $\{\langle$Zen, Down&Out$\rangle$, $\langle$Cuckoo, Zen$\rangle$, $\langle$Spire, Jude$\rangle\}$

It won't even be unusual for there to be infinite number of different interpretations, and there are certainly infinite numbers of different universes.

4 Semantic Values

Together we refer to a universe and some interpretation that maps symbols to things in that universe as a *model*, M:

$$M = \langle \mathcal{U}, \mathcal{I} \rangle$$

Once you have a universe and an interpretation (a model), then you can start to work out the meanings of terms, atoms and wffs with respect to that model. We refer to this as the *semantic value* or *denotation*.

The semantic value of a term, atom or wff α with respect to a model is written:

$$[\![\alpha]\!]^M$$

Here's how to compute the semantic value:

- If α is a constant symbol, function symbol or predicate symbol, then
  $$[\![\alpha]\!]^M = \mathcal{I}(\alpha)$$
  (i.e. the meaning is stipulated by $\mathcal{I}$).
- For any term comprising an n-ary function symbol F applied to terms $T_1, T_2, \ldots, T_n$, then
  $$[\![F(T_1, T_2, \ldots, T_n)]\!]^M = [\![F]\!]^M([\![T_1]\!]^M, [\![T_2]\!]^M, \ldots, [\![T_n]\!]^M)$$
  i.e. you get the function that corresponds to F (as stipulated by $\mathcal{I}$), you work out the objects that correspond to $T_1, T_2, \ldots, T_n$, and then you apply the function to these objects. This gets you an object, and this object is the meaning of the term $F(T_1, T_2, \ldots, T_n)$.
- For any atom comprising an n-ary predicate symbol P applied to terms $T_1, T_2, \ldots, T_n$ then
  $$[\![P(T_1, T_2, \ldots, T_n)]\!]^M = \text{ true iff } \langle [\![T_1]\!]^M, [\![T_2]\!]^M, \ldots, [\![T_n]\!]^M \rangle \in [\![P]\!]^M$$
  i.e. you get the relation that corresponds to P (as stipulated by $\mathcal{I}$), you work out the objects that correspond to $T_1, T_2, \ldots, T_n$, and you see whether this n-tuple of objects is an element of the relation.
- For any quantified wff $\forall XW$, $[\![\forall XW]\!]^M$ is true iff $[\![W]\!]^M$ is true for all assignments of entities in $\mathcal{U}$ to X. Otherwise, it is false.
- For any quantified wff $\exists XW$, $[\![\exists XW]\!]^M$ is true iff $[\![W]\!]^M$ is true for at least one assignment of an entity in $\mathcal{U}$ to X. Otherwise, it is false.
- For any compound wff $\neg W, W_1 \wedge W_2, W_1 \vee W_2, W_1 \Rightarrow W_2$ and $W_1 \Leftrightarrow W_2$, determine $[\![W]\!]^M, [\![W_1]\!]^M$ and $[\![W_2]\!]^M$ and use these truth values as in the usual way to determine the truth value of the compound wff.

Question.

In the lecture, we'll work out the semantic values of the following wffs using $\mathcal{I}_1$:

- $p(b, a)$
- $p(a, b)$
- $q(c, f(b))$
- $\neg p(b, a)$
- $p(b, a) \wedge p(a, b)$
- $p(a, b) \Rightarrow q(c, f(b))$
- $\exists x p(b, x)$
- $\forall x p(b, x)$
- $\exists x \forall y p(x, y)$
- $\forall y \exists x p(x, y)$

And we'll work out the semantic value of the following using $\mathcal{I}_2$:

- $p(b, a)$

5 Value Assignment Functions (Optional)

If you look up the semantics of FOPL in a good textbook, you'll see a more formal treatment than the one I've given. I'm not going to go into the details here. I'll just briefly say what's going on in these more formal treatments.

These more formal treatments use another function, typically called g, referred to as a *value assignment function*. g maps variables to objects. The semantic value of α is then computed with respect to both M and g:

$$[\![\alpha]\!]^{M,g}$$

The upshot of this is:

- it allows a semantic value to be given to wffs that have free variables; and
- it makes the definitions of the semantic values of quantified wffs more properly recursive.

Exercise

Consider a logic using constant symbol a, function symbol f (of arity 1), and predicate symbols p (of arity 1) and q (of arity 2).

Consider a model M, in which $\mathcal{U} = \{1, 2\}$ and $\mathcal{I}$ is defined as follows:

$$\begin{array}{lcl}
\mathcal{I}(a) & = & 1 \\
\mathcal{I}(f) & = & \{1 \mapsto 2, 2 \mapsto 1\}, \text{i.e. } f \text{ corresponds to the function that maps 1 to 2 and vice versa} \\
\mathcal{I}(p) & = & \{\langle 2\rangle\}, \text{i.e. } p \text{ corresponds to the relation which is true of the object 2, but not the object 1} \\
\mathcal{I}(q) & = & \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,2\rangle\}
\end{array}$$

Now evaluate the following:

1. $[\![p(a) \wedge q(a,a)]\!]^M$
2. $[\![p(a) \Rightarrow q(a,a)]\!]^M$
3. $[\![\neg p(f(a)) \vee q(a, f(a))]\!]^M$
4. $[\![\exists x(p(f(x)) \wedge q(x, f(a)))]\!]^M$
5. $[\![\exists x(p(x) \wedge q(x,a))]\!]^M$
6. $[\![\forall x(p(x) \Rightarrow q(f(x),a))]\!]^M$
7. $[\![\forall x \exists y\, q(x,y)]\!]^M$
8. $[\![\exists y \forall x\, q(x,y)]\!]^M$