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An Approach to Robustness in Matching Problems under Ordinal Preferences

Post-viva presentation

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Outline

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   - Matching Problems
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   - Objective
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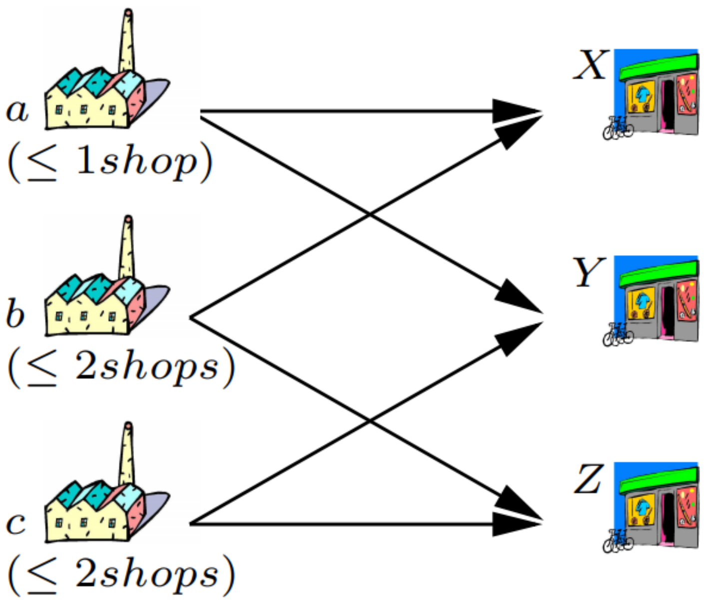
   - Verification of (1,b)-supermatches
   - An approach for (1,1)-supermatches
   - Complexity results
   - Models
3. Robust Stable Roommates Problem
4. Summary
5. Conclusion

Why do we need robustness?

Many problems, especially in the real-world, are usually sensitive to perturbations:
- measurement mistakes,
- errors in data,
- lacking a clear objective,
- unexpected events, etc.

[1]

An Introductory Constraint Programming Example – the Warehouse Allocation Problem

[1] 

(a) Instance

Each shop must be supplied products from at least one of the suitable warehouses!

[1] Emmanuel Hebrard. Robust solutions for constraint satisfaction and optimisation under uncertainty. PhD thesis, University of New South Wales, 2007.

An Introductory Constraint Programming Example – the Warehouse Allocation Problem

Each shop is supplied products from some warehouses.

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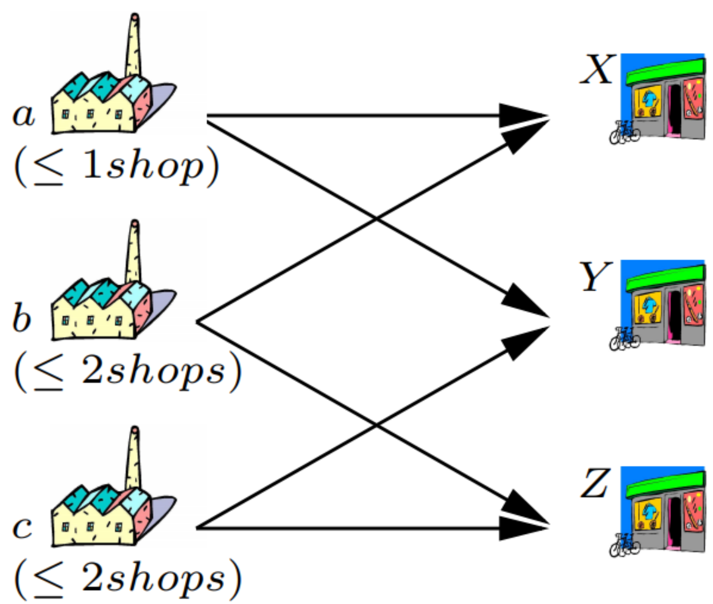


(a) Instance

$$\langle X = a, Y = c, Z = b \rangle$$
$$\langle X = a, Y = c, Z = c \rangle$$
$$\langle X = b, Y = a, Z = b \rangle$$
$$\langle X = b, Y = a, Z = c \rangle$$
$$\langle \underline{X = b}, \underline{Y = c}, \underline{Z = b} \rangle$$
$$\langle X = b, Y = c, Z = c \rangle$$

(b) Solutions

[1] Emmanuel Hebrard. Robust solutions for constraint satisfaction and optimisation under uncertainty. PhD thesis, University of New South Wales, 2007.

Some Existing Robustness Notions

Robustness has many different definitions in Robust Optimization.

Robustness in CP and SAT

- *Climent et al.: "a robust solution has a high probability to remain solution after changes in the environment."* [1]

- *Handbook of CP: "a robust solution is likely to remain solution even after the change has occurred, or to need only minor repairs."* [2]

[1] Laura Climent, Richard J. Wallace, Miguel A. Salido, and Federico Barber. Robustness and stability in constraint programming under dynamism and uncertainty. *J. Artif. Intell. Res.*, 49:49–78, 2014.
[2] Handbook of Constraint Programming. Francesca Rossi, Peter van Beek, and Toby Walsh (Eds.). Elsevier Science Inc., New York, NY, USA, 2006.

Robustness using (a,b)-models

(a,b)-supermodels [1] - SAT

An (a,b)-supermodel is a model such that if we modify the values taken by the variables in a set of size at most a (breakage), another model can be obtained by modifying the values of the variables in a disjoint set of size at most b (repair).

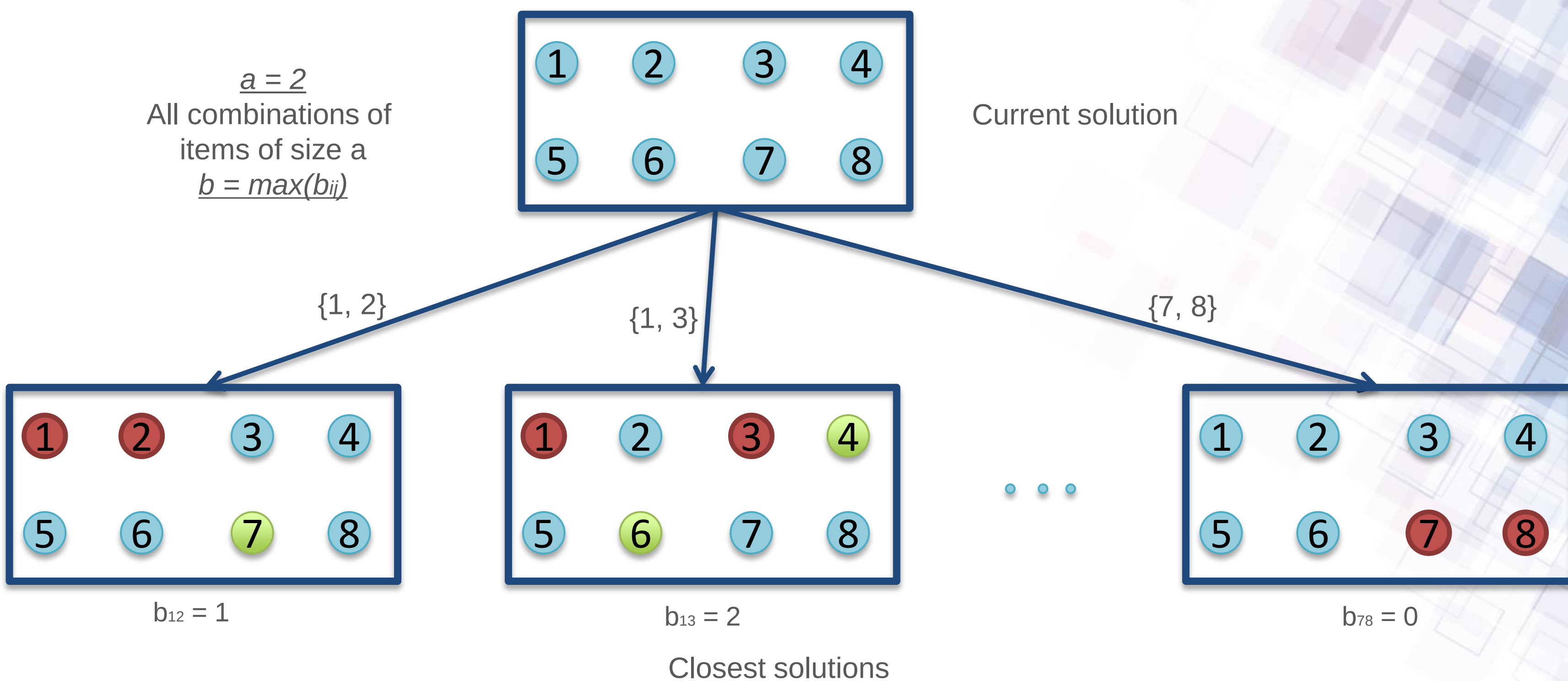
(a,b)-supersolutions [2] - CP

An (*a*,*b*)-super solution is a solution which if any *a* variables break, the solution can be repaired by providing repair by changing a maximum of *b* other variables.

[1] Matthew L. Ginsberg, Andrew J. Parkes, and Amitabha Roy. Supermodels and robustness. In *In AAAI/IAAI*, pages 334–339, 1998.
[2] Emmanuel Hebrard, Brahim Hnich, and Toby Walsh. Robust solutions for constraint satisfaction and optimization. In Proceedings of ECAI'2004, Valencia, Spain, August 22-27, 2004, pages 186–190, 2004.

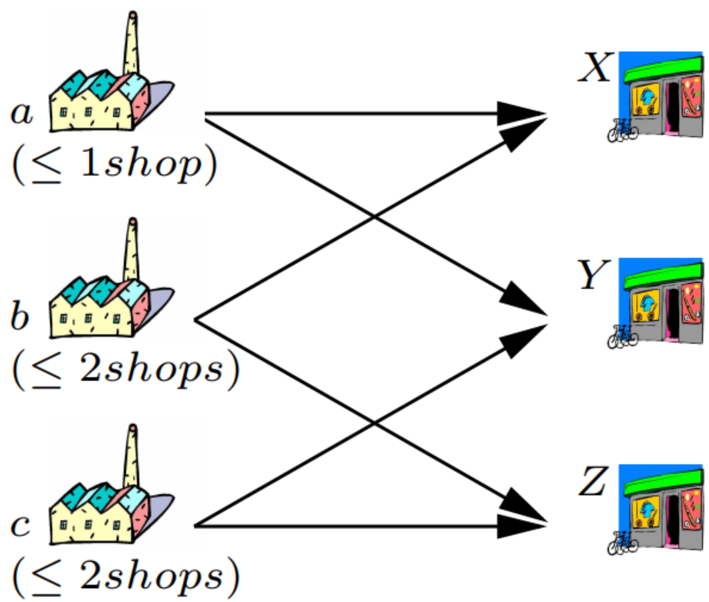
Robustness using (a,b)-models

(a,b)-model



← Introductory CP Example

Some solutions are more robust than others!



(a) Instance

X can not be supplied from *a* anymore. Thus, it must be supplied from *b.*

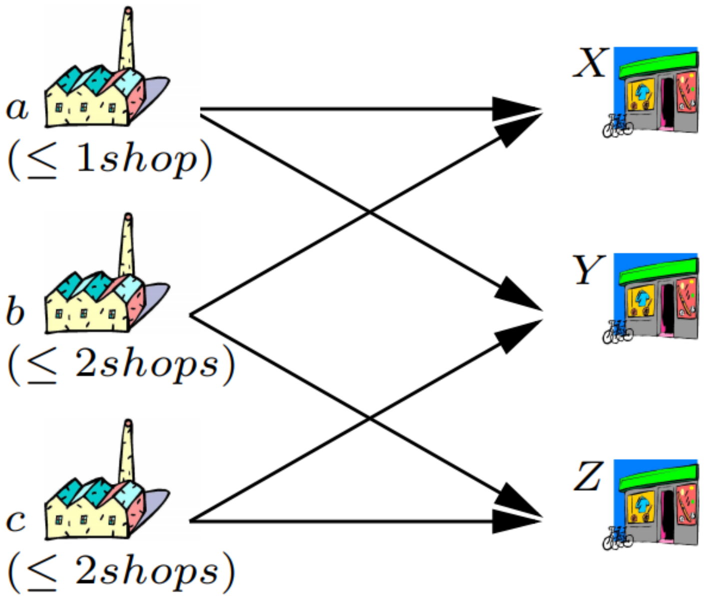
(X=a) → 0

$$\boxed{\langle X=a, Y=c, Z=b\rangle}$$
$$\cancel{\langle X=a, Y=c, Z=c\rangle}$$
$$\langle X=b, Y=a, Z=b\rangle$$
$$\langle X=b, Y=a, Z=c\rangle$$
$$\boxed{\langle \underline{X=b}, \underline{Y=c}, \underline{Z=b}\rangle}$$
$$\langle X=b, Y=c, Z=c\rangle$$

(b) Solutions

← Introductory CP Example

Some solutions are more robust than others!



(a) Instance

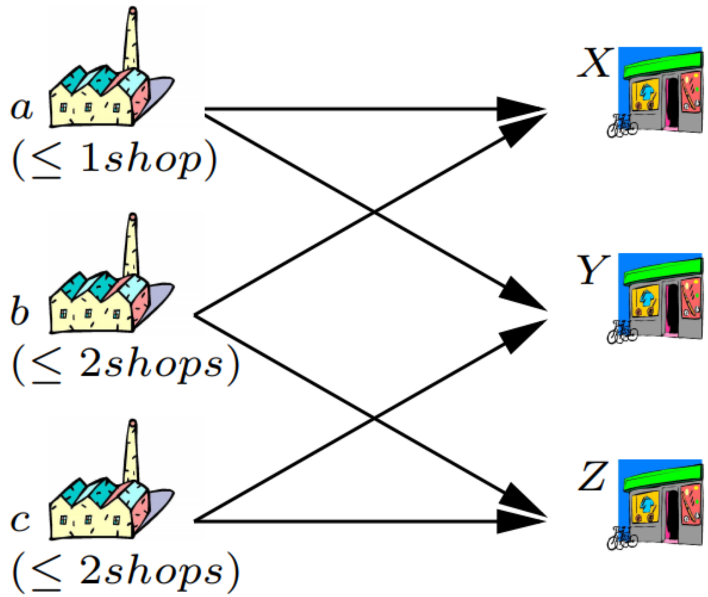
(X=a) → 0, (Y=c) → 1,

$$\boxed{\langle X = a, Y = c, Z = b \rangle}$$
$$\cancel{\langle X = a, Y = c, Z = c \rangle}$$
$$\boxed{\langle X = b, Y = a, Z = b \rangle}$$
$$\langle X = b, Y = a, Z = c \rangle$$
$$\cancel{\langle X = b, Y = c, Z = b \rangle}$$
$$\cancel{\langle X = b, Y = c, Z = c \rangle}$$

(b) Solutions

← Introductory CP Example

Some solutions are more robust than others!



(a) Instance

(X=a) → 0, (Y=c) → 1, (Z=b) → 0
(1,1)-super solution

$\boxed{\langle X = a, Y = c, Z = b \rangle}$

$\boxed{\langle X = a, Y = c, Z = c \rangle}$

$\cancel{\langle X = b, Y = a, Z = b \rangle}$

$\langle X = b, Y = a, Z = c \rangle$

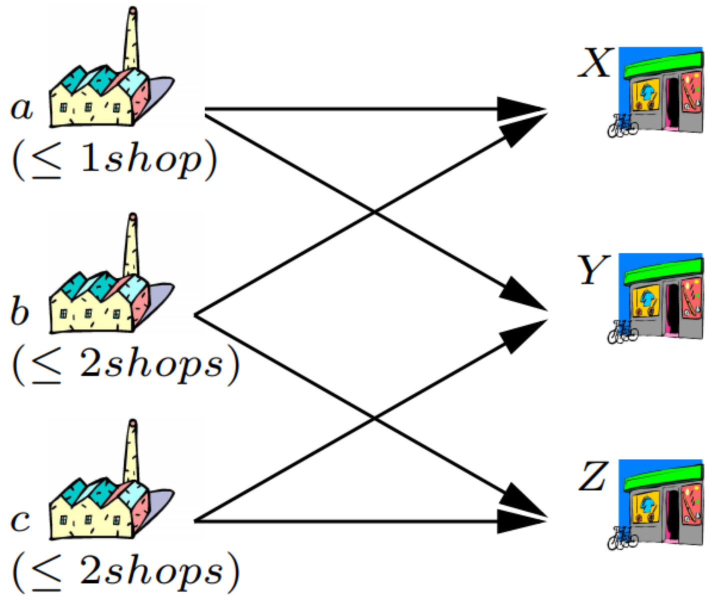
$\cancel{\langle X = b, Y = c, Z = b \rangle}$

$\langle X = b, Y = c, Z = c \rangle$

(b) Solutions

← Introductory CP Example

Some solutions are more robust than others!



(X=a) → 0, (Y=c) → 0, (Z=b) → 0
(1,0)-super solution

$\langle X = a, Y = c, Z = b \rangle$ X=a

$\langle X = a, Y = c, Z = c \rangle$

$\langle X = b, Y = a, Z = b \rangle$ Y=c

$\langle X = b, Y = a, Z = c \rangle$

$\boxed{\langle \underline{X = b}, \underline{Y = c}, \underline{Z = b} \rangle}$

$\langle X = b, Y = c, Z = c \rangle$ Z=b

(b) Solutions

A_5 is a more robust solution than A_1 in case of an unforeseen event!

Matching under Ordinal Preferences

Goal: Find a matching between some agents respecting some optimality criteria.

Example problems include:

- Hospitals/Resident (HR),
- Stable Marriage (SM),
- Stable Roommates (SR),
- Kidney Exchange,
- Ride Sharing, etc.

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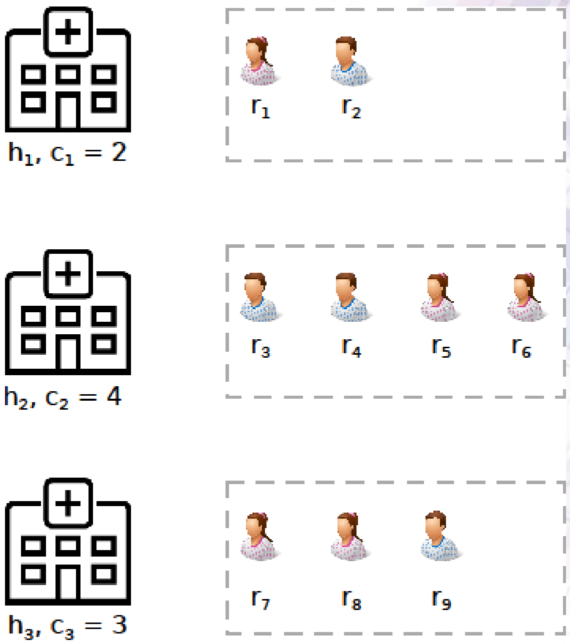
Stable



An HR instance of 3 hospitals and 9 residents.

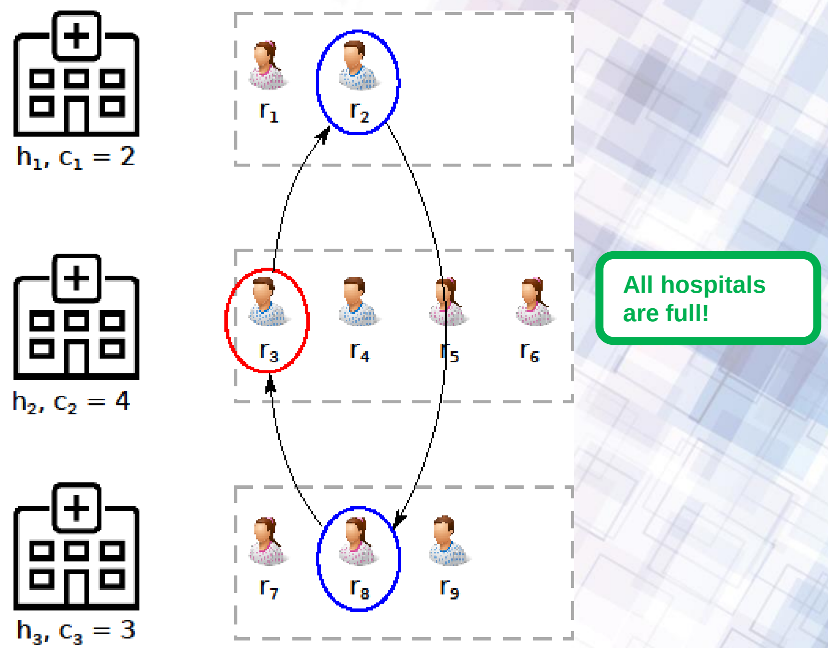
Motivation

Goal: Find a matching between some agents respecting some optimality criteria.

Example problems include:

- Hospitals/Resident (HR),
- Stable Marriage (SM),
- Stable Roommates (SR),
- Kidney Exchange,
- Ride Sharing, etc.

Stable and Robust?



Resident r_3 must be relocated due to an unforeseen event.

Matching under Ordinal Preferences

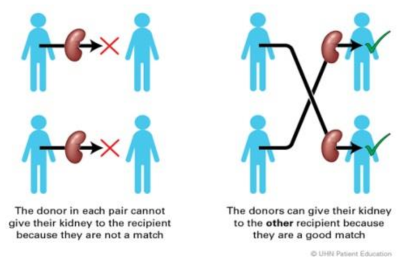
Goal: Find a matching between some agents respecting some optimality criteria.

Example problems include:

- Hospitals/Resident (HR),
- Stable Marriage (SM),
- Stable Roommates (SR),
- Kidney Exchange,
- Ride Sharing, etc.

- Migration of virtual machines in Cloud Computing,
- Content delivery on the Internet,
- Wireless resource management, etc.

- Peer-to-peer networks (P2P),
- File sharing (torrent)



Thesis Objective

Motivation

Need **robustness + stability** in matching problems to handle *unexpected events*.

Thesis

Achieving both stability and robustness is possible.

Proposal

A new notion: **(a,b)-supermatches** = (robust + stable) matching.

Stable Marriage Problem (SM)

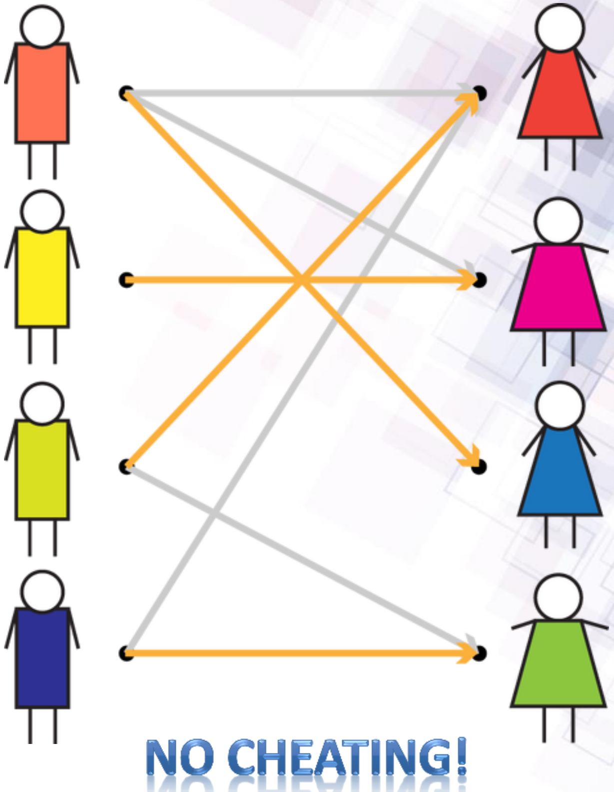
A specific case of the HR with capacities = 1.

Input

- A set of men,
- A set of women,
- Strictly ordered preference lists of both:
  - men over women,
  - women over men.

Output

A stable matching such that everyone is matched to a person and no unmatched pairs prefer each other to their partners.

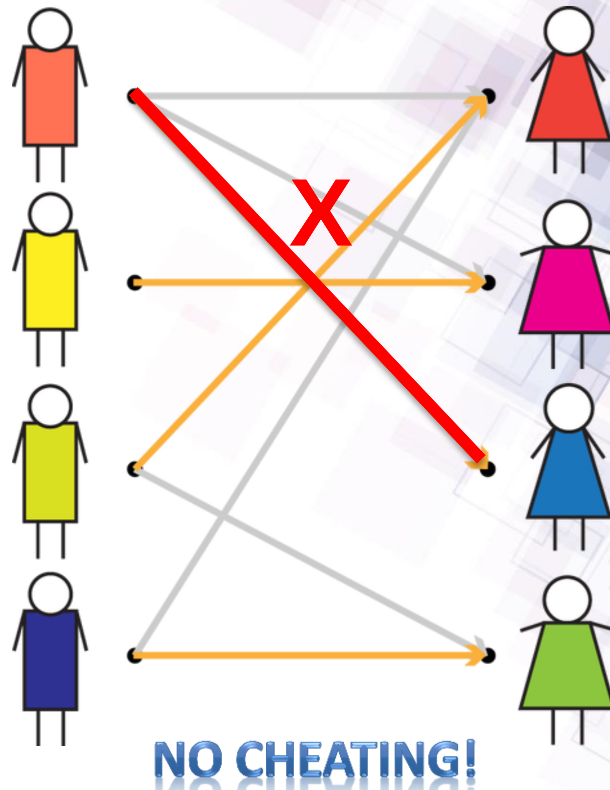


Stable Marriage Problem (SM)

A specific case of the HR with capacities = 1.

What if a couple must break-up?

✓ *Find alternative partners to them. (break-up some other pairs)*



(a,b)-supermatches

An (*a*,*b*)-supermatch is a matching between the agents that is both stable and robust subject to some additional constraints.

(*a*, *b*)-supermatch

A stable matching such that if *any combination of **a** pairs* want to leave the matching, there exists an *alternative matching* in which those *a* pairs are assigned new partners, and in order to obtain the new assignment *at most **b** other pairs* are broken.

(1,b)-supermatches: A restricted case, where a = 1.

(1,1)-supermatches: A *very* restricted case, where a = 1 and b = 1.

Verifying if a matching is a (1,b)-supermatch

Given a stable matching (e.g. M = {(Bob, Arya), (Mike, Asha), (Tom, Cathy)})
Question: Is M a (1,b)-supermatch?

➢ We proposed a polynomial-time procedure that uses the properties of *rotation posets*.

Procedure outline

- Find the **closest stable matchings to M**:
  - ✓ M_1 when (Bob, Arya) breaks up.
  - ✓ M_2 when (Mike, Asha) breaks up.
  - ✓ M_3 when (Tom, Cathy) breaks up.
- **Max(M_1, M_2, M_3)** sets the value of **b**.

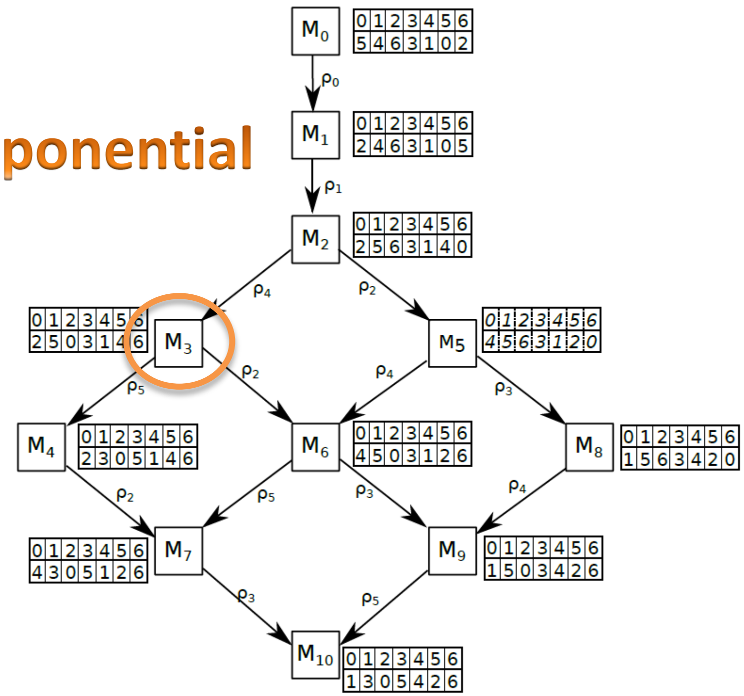
Publication
Begum Genc, Mohamed Siala, Gilles Simonin, Barry O'Sullivan: *Robust Stable Marriage*. **AAAI 2017, AAAI Press**: 4925-4926

SM structure

➢ 1-1: Closed subsets & Stable matchings

Lattice of Stable Matchings

Exponential

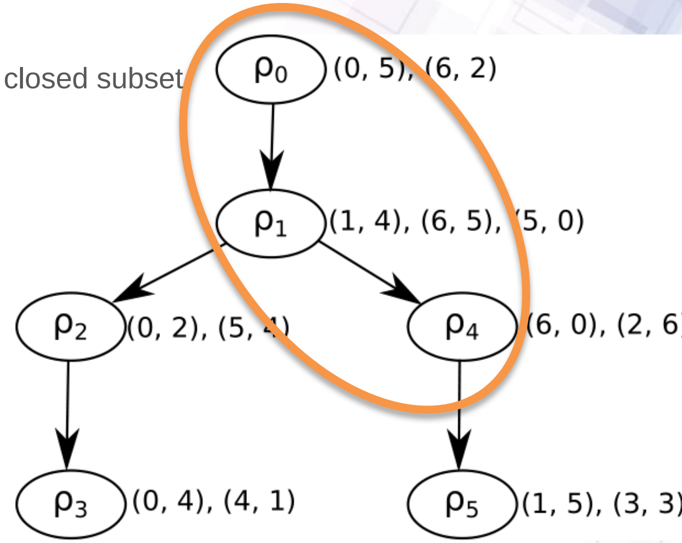


Preference lists

m_0	0 6 5 2 4 1 3
m_1	6 1 4 5 0 2 3
m_2	6 0 3 1 5 4 2
m_3	3 2 0 1 4 6 5
m_4	1 2 0 3 4 5 6
m_5	6 1 0 3 5 4 2
m_6	2 5 0 6 4 3 1

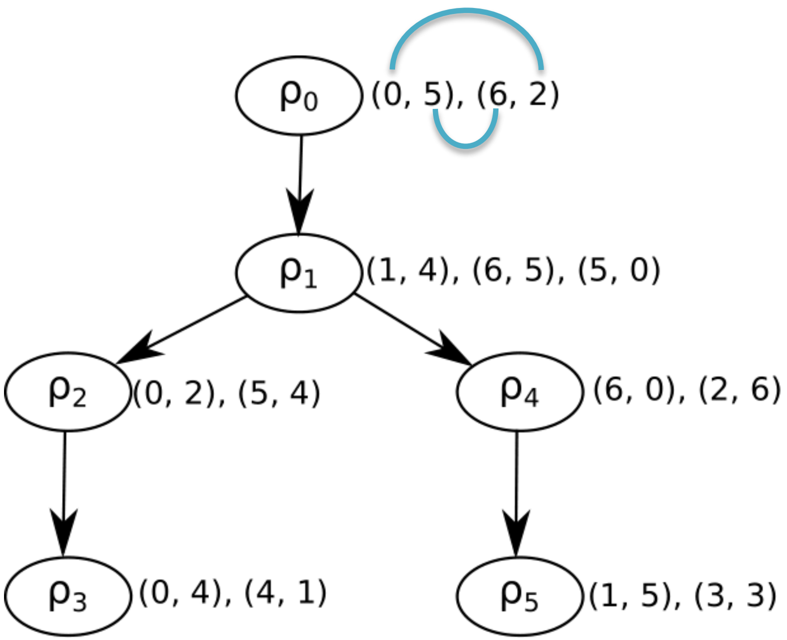
w_0	2 1 6 4 5 3 0
w_1	0 4 3 5 2 6 1
w_2	2 5 0 4 3 1 6
w_3	6 1 2 3 4 0 5
w_4	4 6 0 5 3 1 2
w_5	3 1 2 6 5 4 0
w_6	4 6 2 1 3 0 5

A closed subset



Rotation Poset

Rotations



ρ_0

Eliminates pairs : <0, 5>, <6, 2>
Produces pairs : <0, 2>, <6, 5>

ρ_4

Eliminates pairs : <6, 0>, <2, 6>
Produces pairs : <6, 6>, <2, 0>

For each pair, there exists at most 1 production rotation and 1 elimination rotation.

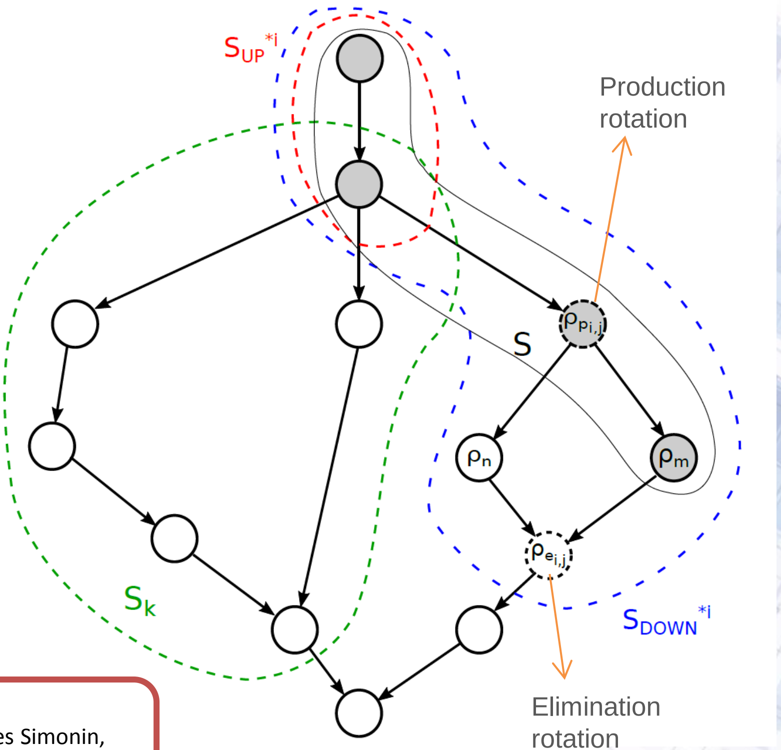
Illustration of the procedure for (1,b)-supermatches

S: Corresponds to the closed subset of the given stable matching ***M***.

S_{UP}^{*i}: First potential closest stable matching to **S** when man *i* and his partner leaves ***M***.

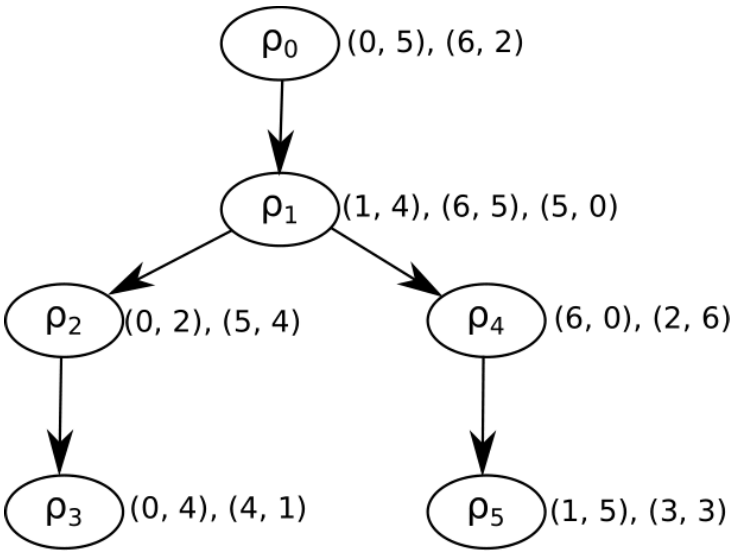
S_{DOWN}^{*i}: Second potential closest stable matching to **S** when man *i* and his partner leaves ***M***.

S_k: No other stable matchings can be closer to **S** than S_{UP}^{*i} or S_{DOWN}^{*i}.

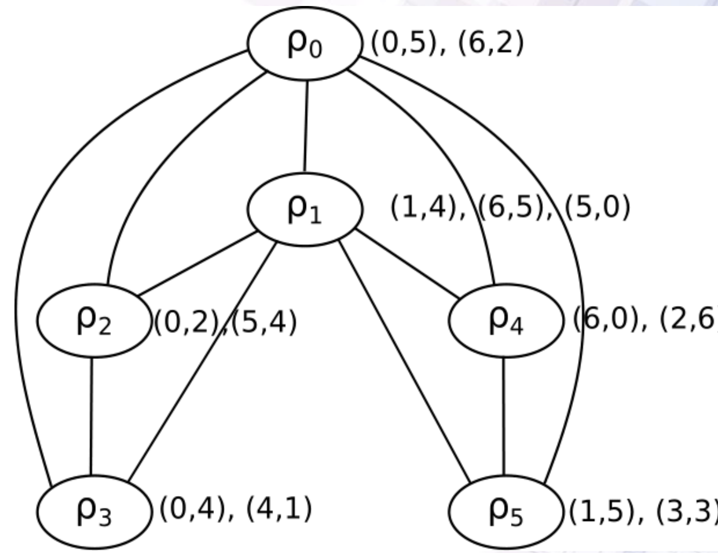


Publication
Begum Genc, Mohamed Siala, Gilles Simonin, Barry O'Sullivan: *Finding Robust Solutions to Stable Marriage*. **IJCAI 2017**: 631-637

Complexity… A model for identifying (1,1)-supermatches using independent sets



A sample rotation poset G.



Transitive version of G.

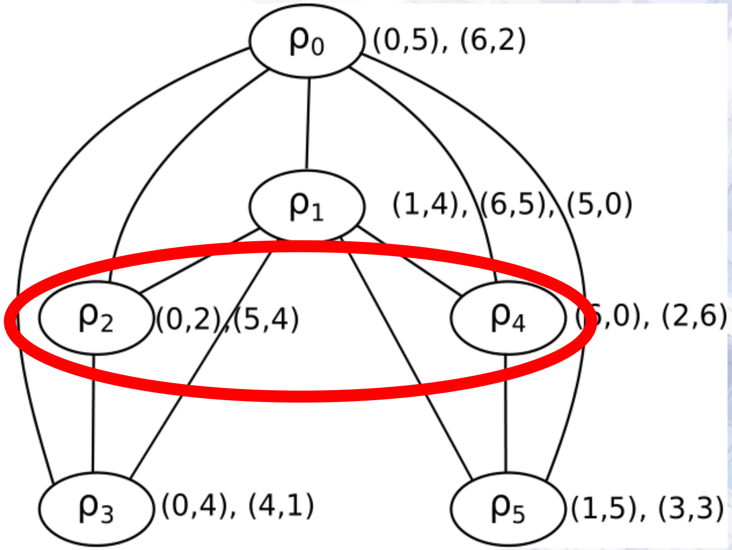
The set of non-fixed men = {0, 1, 2, 3, 4, 5, 6}

A model for identifying (1,1)-supermatches using independent sets

Find an ***I*** such that:

- ***I* U neighbours** covers all non-fixed men in their rotations of size 2.

Any such ***I*** corresponds to a unique (1,1)-supermatch ***M***.



Independent set *I*

Publication
Begum Genc, Mohamed Siala, Gilles Simonin, Barry O'Sullivan: *Complexity Study for the Robust Stable Marriage Problem.* ***Theoretical Computer Science 775, Elsevier:*** *76-92* ***(2019)***

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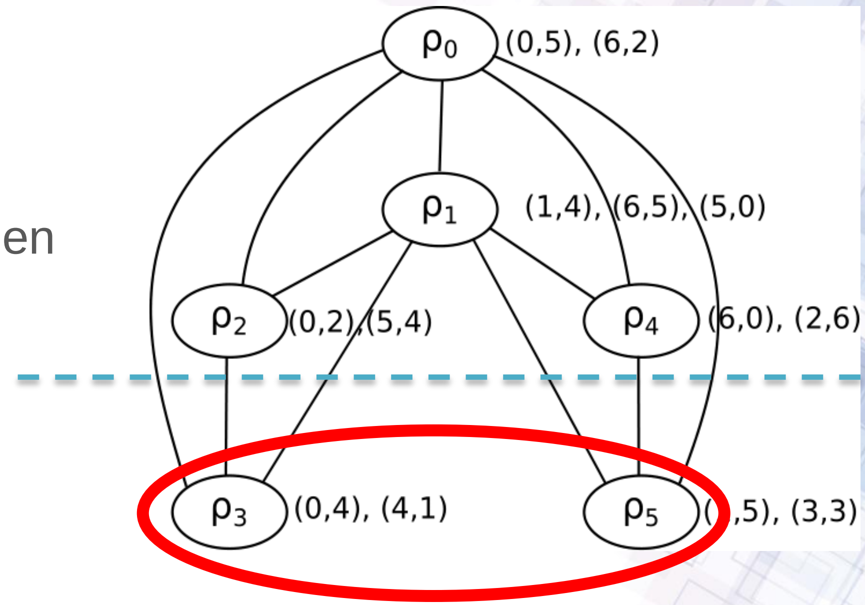

Neighbours of *I*

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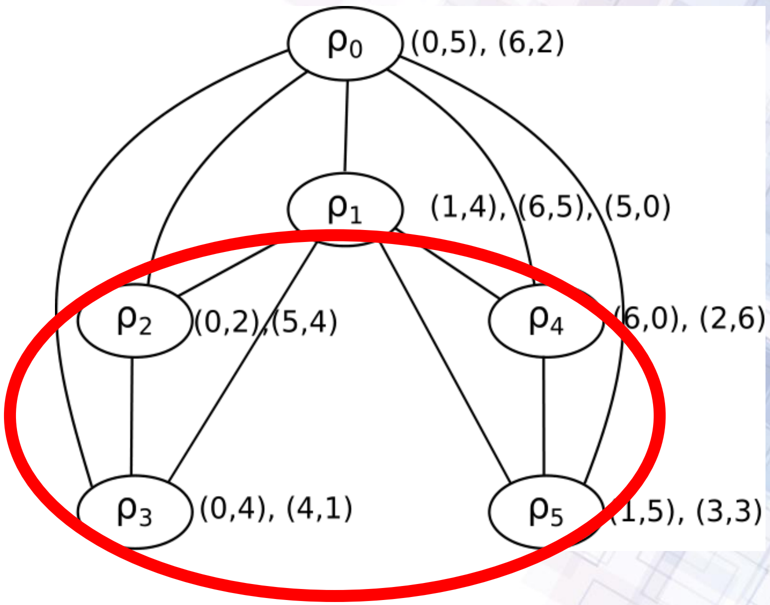

***I* U neighbours = {0, 1, 2, 3, 4, 5, 6}**

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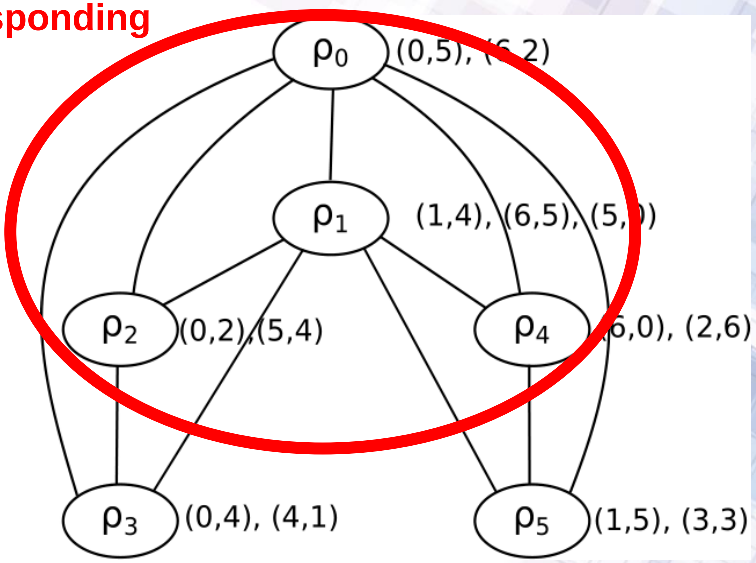
Begum Genc, Mohamed Siala, Gilles Simonin, Barry O'Sullivan: *Complexity Study for the Robust Stable Marriage Problem.* ***Theoretical Computer Science 775, Elsevier:*** *76-92* ***(2019)***

An alternative model for (1,1)-supermatches using independent sets

Closed subset corresponding to *I*



ρ_2 and ρ_4 define a stable matching ***M***

that corresponds to the closed subset

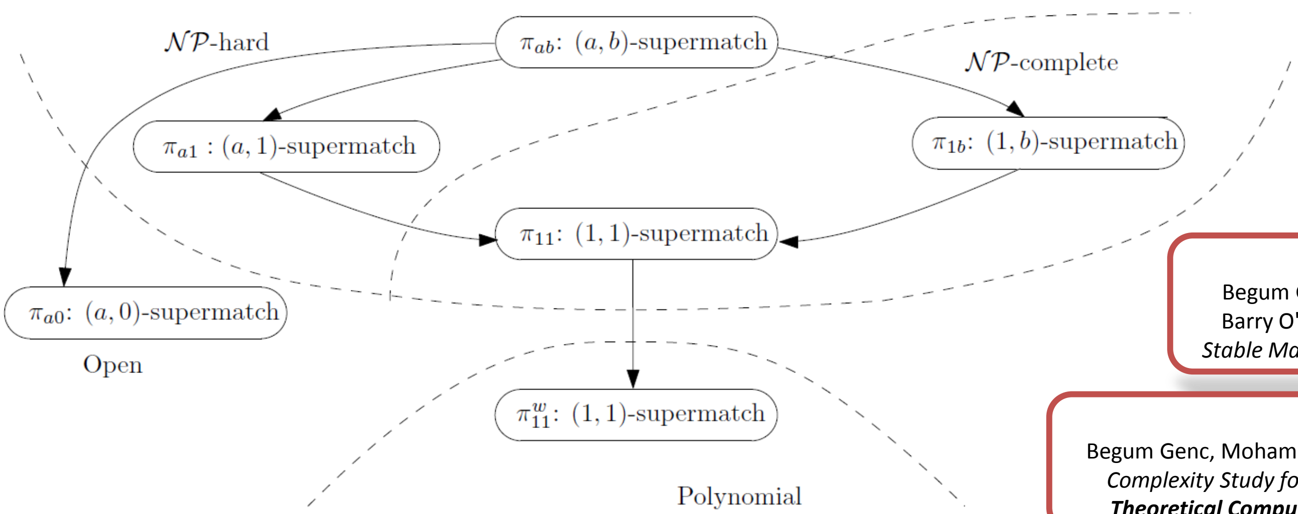
$\boldsymbol{S} = \{\rho_0, \rho_1, \rho_2, \rho_4\}$

Publication

Begum Genc, Mohamed Siala, Gilles Simonin, Barry O'Sullivan: *Complexity Study for the Robust Stable Marriage Problem.* ***Theoretical Computer Science 775, Elsevier:*** *76-92* ***(2019)***

Complexity of RSM

- A special case of SAT (**SAT-SM**) is defined.
- Showed that **SAT-SM is NP-complete** by *Schaefer's Dichotomy Theorem*.
- Showed **equivalency** between SAT-SSM and deciding if there exists a (1,1)-supermatch to a given RSM instance.



Publication
Begum Genc, Mohamed Siala, Gilles Simonin, Barry O'Sullivan: *On the Complexity of Robust Stable Marriage.* ***COCOA 2017, Springer***: *441-448*

Publication
Begum Genc, Mohamed Siala, Gilles Simonin, Barry O'Sullivan: *Complexity Study for the Robust Stable Marriage Problem.* ***Theoretical Computer Science 775, Elsevier:*** *76-92* ***(2019)***

Models to find a (1,b)-supermatch to RSM

- **CP**

  Formulated stable matchings using rotations.

  The aim is to compute the rotations between M and its closest stable matchings for each pair.

- **Local Search**

  Start from a random stable matching M.

  Explore the neighbours of M.

- **Genetic Algorithm**

  Start from a *random population* of stable matchings.

  Evolve the population by applying crossovers and mutations.

- **Genetic Local Search**

  Start from a *random population*.

  Explore the neighbours of the products of crossover.

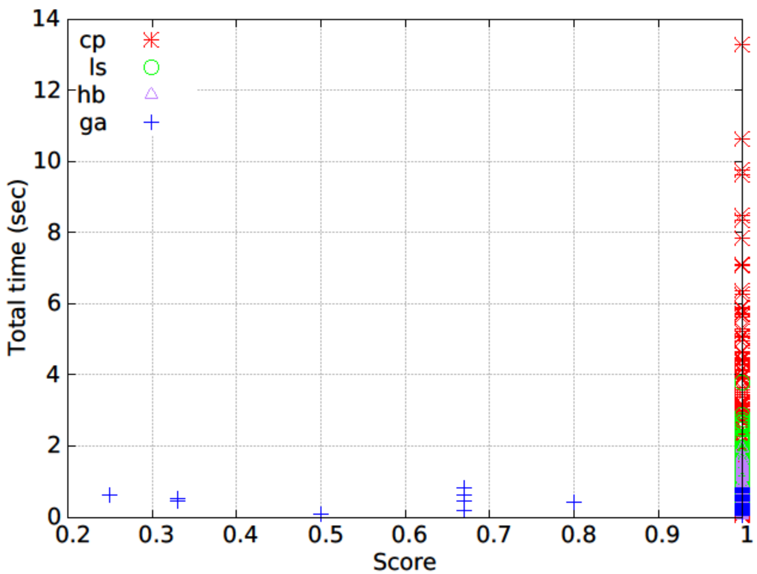
All based on the polynomial-time procedure

Publication
Begum Genc, Mohamed Siala, Gilles Simonin, Barry O'Sullivan: *Finding Robust Solutions to Stable Marriage*. **IJCAI 2017**: 631-637

Model Comparison on Random Instances

$n \in \{50 \times k \mid k \in \{1, . . . , 6\}\}$

Search efficiency on small random instances.

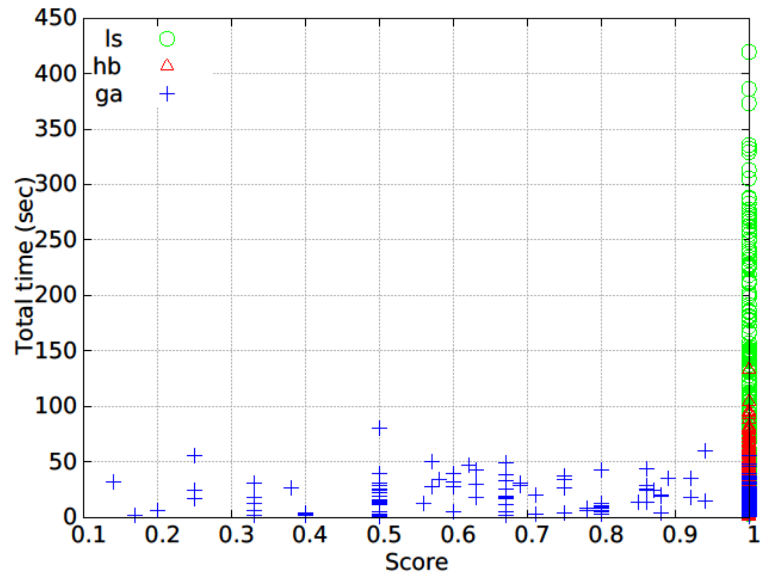


Slower

Smaller b values
(more robust)

$n \in \{100 \times k \mid k \in \{4, . . . , 20\}\}$

Search efficiency on large random instances.



Stable Roommates Problem (SR)

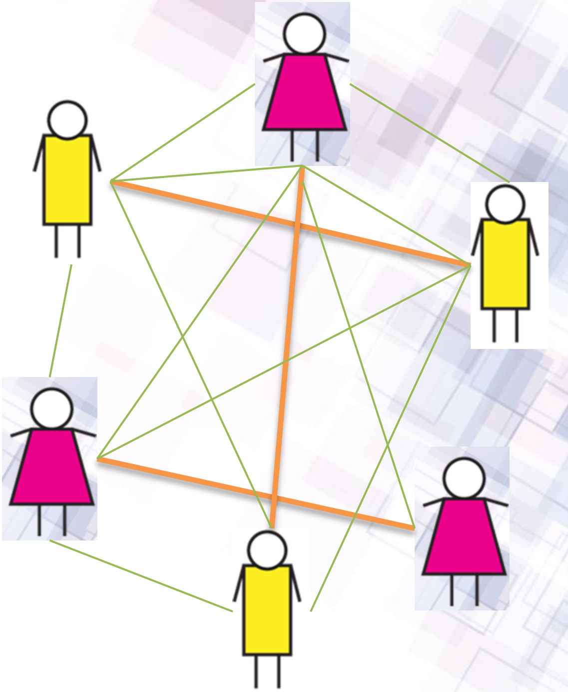
Generalization of the SM, where the sex factor is eliminated.

Input

- A set of people,
- Strictly ordered preference lists of each person over the others.

Output

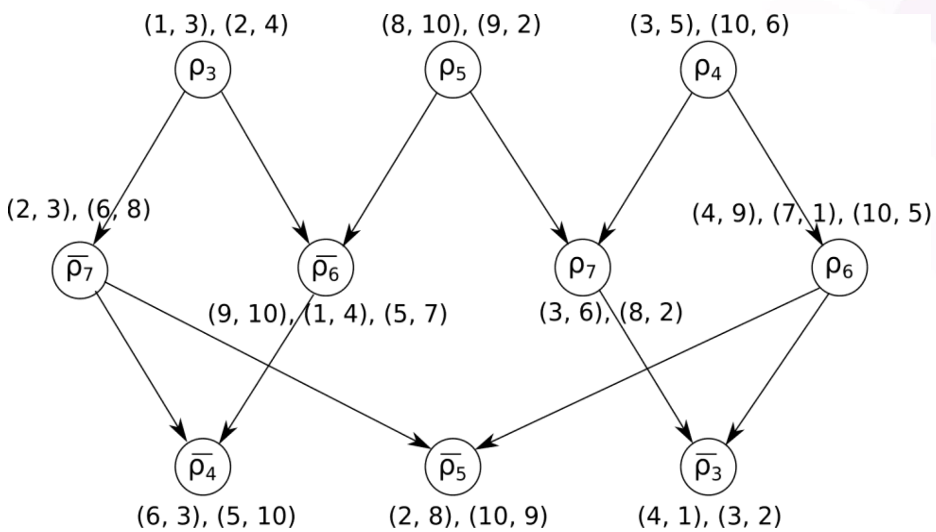
A stable matching such that no unmatched pairs prefer each other to their partners and everyone has a partner.



Robust Stable Roommates Problem (RSR)

RSR is NP-hard!

- The structure is different to the rotation poset of the SM.
- 1-1: Complete closed subsets & Stable matchings



Rotation
$\rho_3 = (1,3), (2,4)$

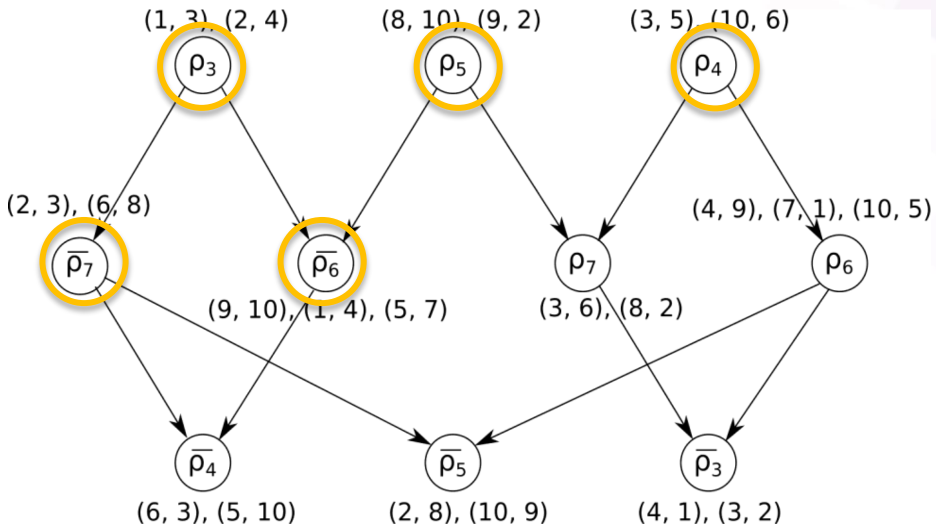
Dual of ρ3
$\overline{\rho_3} = (4,1), (3,2)$

Robust Stable Roommates Problem (RSR)

RSR is NP-hard!

- The structure is different to the rotation poset of the SM.
- 1-1: Complete closed subsets & Stable matchings

Complete closed subset
$S = \{\rho_3, \rho_4, \rho_5, -\rho_6, -\rho_7\}$



Robust Stable Roommates Problem (RSR)

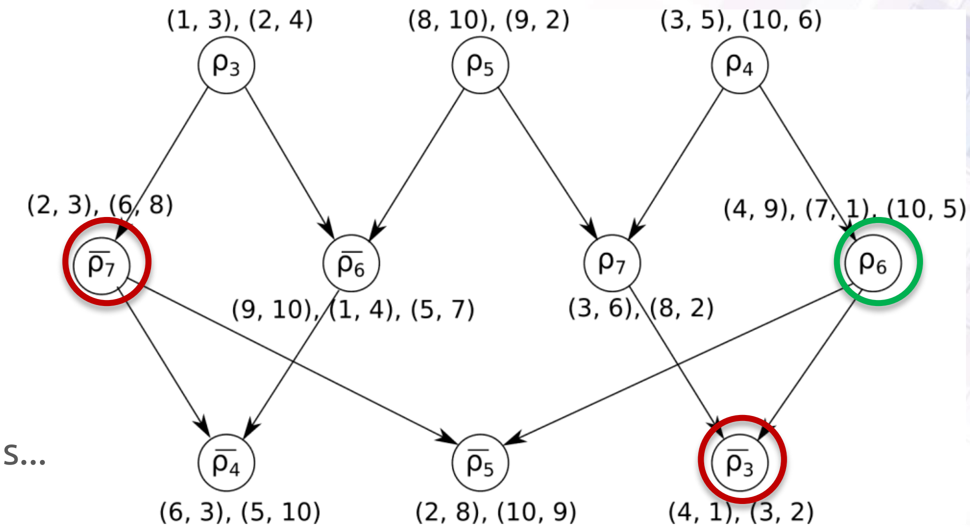
RSR is NP-hard!

There may be up to 2 production or elimination rotations for a pair!

Elimination rotation for pair **{1,7}**
ρ_6

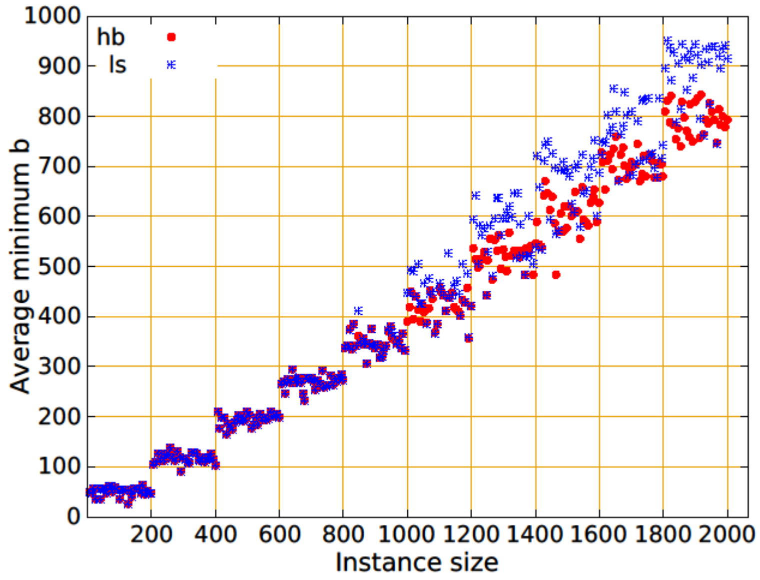
Elimination rotation for pair **{2,3}**
$-\rho_7$ and $-\rho_3$

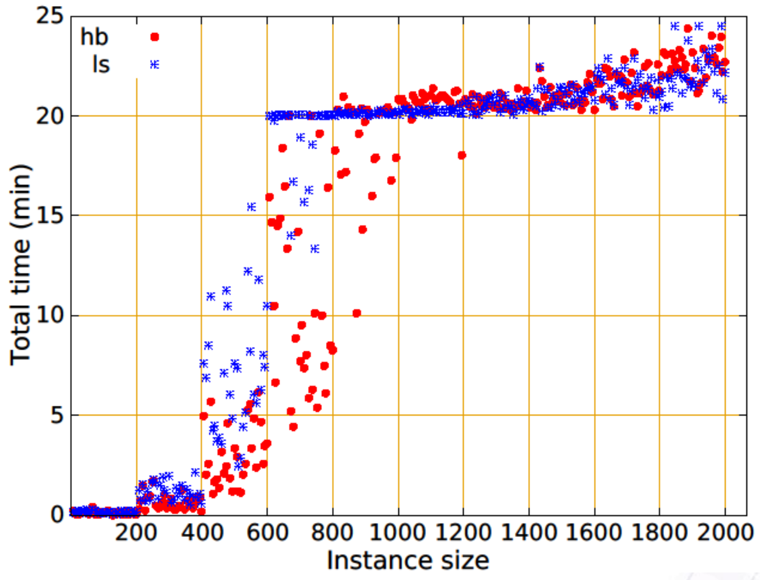
Similar for the production rotations…



Models to find the most robust (1,b)-supermatch

- Local Search (ls)
- Genetic Local Search (hb)





Publication
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MANY – rich in stable matchings

Table 4.6: An SM instance of size 8 that belongs to the original family described by Irving and Leather [IL86a].

	Preference lists of men							
m_1	1	2	3	4	5	6	7	8
m_2	2	1	4	3	6	5	8	7
m_3	3	4	1	2	7	8	5	6
m_4	4	3	2	1	8	7	6	5
m_5	5	6	7	8	1	2	3	4
m_6	6	5	8	7	2	1	4	3
m_7	7	8	5	6	3	4	1	2
m_8	8	7	6	5	4	3	2	1

	Preference lists of women							
w_1	8	7	6	5	4	3	2	1
w_2	7	8	5	6	3	4	1	2
w_3	6	5	8	7	2	1	4	3
w_4	5	6	7	8	1	2	3	4
w_5	4	3	2	1	8	7	6	5
w_6	3	4	1	2	7	8	5	6
w_7	2	1	4	3	6	5	8	7
w_8	1	2	3	4	5	6	7	8

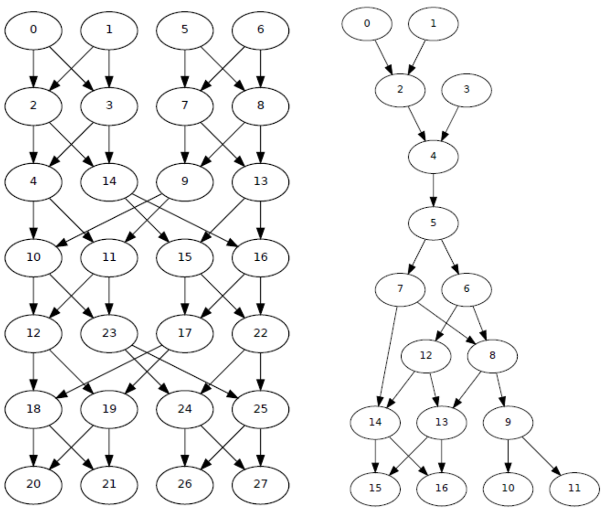
modify

Table 4.7: An SM instance of size 8 that belongs to our benchmark MANY obtained by the original instance given in Table 4.6.

	Preference lists of men							
m_1	1	2	3	4	5	6	7	8
m_2	2	8	4	3	6	5	1	7
m_3	3	4	1	2	7	8	5	6
m_4	4	3	2	1	8	7	6	5
m_5	5	6	7	8	1	2	3	4
m_6	6	5	8	7	2	1	4	3
m_7	7	8	5	6	3	4	1	2
m_8	8	7	5	6	4	3	2	1

	Preference lists of women							
w_1	8	7	6	5	4	3	2	1
w_2	7	8	5	4	3	6	1	2
w_3	6	5	8	7	2	1	4	3
w_4	5	6	7	8	1	2	3	4
w_5	4	3	2	1	8	7	6	5
w_6	3	4	1	2	5	8	7	6
w_7	2	1	4	3	6	5	8	7
w_8	1	2	3	4	5	6	7	8

Very rich in (1,1)-supermatches



(a) Instance given in Table 4.6. (b) Instance given in Table 4.7.

Figure 4.9: Rotation posets corresponding to the large instances.

Summary of Empirical Results

- Hybrid model is able to find stable matchings with low **b** values in large instances. However, it is achieving this by taking advantage of its randomness.
- Local search model is very competitive with the hybrid model.
- Our version of genetic algorithm gets stuck in the local optima.
- We identified a family of SM and SR instances that are very rich in stable matchings. The rich instances often contain (1,1)-supermatches.
- The random RSM instances are very consistent to little modifications in their preference lists in terms of their robustness. The random RSR instances are not.

Summary of Contributions

- ✓ A novel notion of robustness that uses fault-tolerance for matchings under ordinal preferences.
- ✓ Polynomial-time procedures for deciding (1,b)-supermatches.
- ✓ Complexity study for finding (a,b)-supermatches.
- ✓ Identification of structural properties for the SM and the SR.
- ✓ A number of different models to solve the problem.
- ✓ Open problems.
- ✓ A new public dataset.

Future Directions

1. There are several fields that we left as open problems in terms of the complexity.

Problem	$\mathcal{NP}$-hard	$\mathcal{NP}$-complete	$\mathcal{P}$
$(1,1)$-supermatch $(\boldsymbol{\pi}_{11})$	✓	✓	x
$(1,1)$-supermatch from family $\mathrm{F}_w(\boldsymbol{\pi}^w_{11})$	x	x	✓
$(1,b)$-supermatch $(\boldsymbol{\pi}_{1b})$	✓	✓	x
(a,b)-supermatch $(\boldsymbol{\pi}_{ab})$	✓	?	x
$(a,0)$-supermatch $(\boldsymbol{\pi}_{a0})$	?	?	?
$(a,1)$-supermatch $(\boldsymbol{\pi}_{a1})$	✓	?	x

2. Different, fast models can be developed.
3. Current models can be improved.
4. Experiments can be made using real-world data.
5. (a,b)-supermatches for other matching problems can be studied.

Already getting some attention!

1. There are several fields that we left as open problems in terms of the complexity.

→ K Cechlárová, A Cseh, D Manlove, ***Selected open problems in matching under preferences***, Bulletin of EATCS, 2019

(1, 1)-supermatch from family $F_w(\pi^w_{11})$ | x | x | ✓

Genc et al. [42] showed that the problem of deciding if there exists a $(1, b)$-supermatch is NP-complete in SMI for any $b \geq 1$, which also implies that this problem is NP-hard in SRI. However, for the more general case of (a, b)-supermatches, it is not even known whether the problem belongs to NP. By contrast, given a stable matching M in an SRI instance, Genc et al. [41] gave a polynomial-time algorithm to verify whether M is a $(1, b)$-supermatch. The algorithm uses a deep knowledge of the structure of the set of all stable matchings, described by the complete closed subsets of the reduced rotation poset of the given SRI instance.

2. D
3. C
4. E
5. (a,b)-supermatches for other matching problems can be studied.

Conclusion

It is possible to achieve both robustness and stability in matching problems.

Reference List

All authored by **Begum Genc, Mohamed Siala, Gilles Simonin, Barry O'Sullivan**

Journals

- Complexity Study for the Robust Stable Marriage Problem. ***Theoretical Computer Science 2019***, Elsevier: 76-92

Conferences

- Robust Stable Marriage. ***AAAI 2017***, AAAI Press: 4925-4926
- On the Complexity of Robust Stable Marriage. ***COCOA 2017***, Springer: 441-448
- Finding Robust Solutions to Stable Marriage. ***IJCAI 2017***: 631-637
- An Approach to Robustness in the Stable Roommates Problem and its Comparison with the Stable Marriage Problem. ***CPAIOR 2019***, Springer: 320-336

Others

- Finding Robust Solutions to Stable Marriage. CoRR abs/1705.09218 (2017)
- On the Complexity of Robust Stable Marriage. CoRR abs/1709.06172 (2017)